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# Optimal and predefined policies for the static lot sizing problem in a two stage recovery system

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**Abstract**

Analyzing static lot sizing problems has always attracted a considerable interest in scientific literature. A commonly applied methodology to solve the trade-off between setup and holding costs is to order the Economic Order Quantity (EOQ) whenever the corresponding inventory is depleted. Yet, this simple proceeding can only be applied as long as there is only a single source of supply. Recovery systems, however, obtain in general two sources of supply, remanufacturing product returns and fabricating new products. Therefore, a more sophisticated approach needs to be taken into account for this kind of problem setting. This contribution focusses on extending the current knowledge in this field of research by showing that non-equal remanufacturing batches propose a significant cost reduction for some parameter classes. Furthermore, a more general optimization approach is introduced that allows to evaluate the solution quality of the preset policy structures.

## 1 Introduction

The growing environmental concern of their customers combined with an increasing price consciousness poses a challenging task for many manufacturing companies. This development in customer behavior supports the manufacturing companies to consider product recovery as a viable alternative to satisfy customer demand. Depending on the degree of disassembly and material reuse, Thierry et al. (1995) classify five different recovery options. Among these options, remanufacturing returned products seems to be of special interest since it addresses both issues demanded by their customers. On the one hand, remanufacturing a returned product reduces landfill space as it needs not

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to be disposed of. On the other hand, as a part of the value embedded in the product is saved, the manufacturer is able to offer his customers a significant price discount on the remanufactured product. When accepting this offer, the customer does not face a disadvantage compared to buying a new product since in general the same warranty is issued for both.

In literature, a variety of real-life industrial applications for remanufacturing has been presented ranging from car engines (as in Seitz and Wells, 2006) over photocopiers (as in Thierry et al., 1995) to water pumps for diesel engines (as in Tang and Teunter, 2006). Common to all industrial applications is that remanufacturing a returned product requires a large number of different processing operations. After return, each product is disassembled to obtain its components. All components are inspected whether they can be reused or not. If necessary, mechanical rework processes ensure the required quality standards. Complemented by new components, the remanufactured components are assembled into remanufactured products which can be offered for sale.

For establishing an efficient remanufacturing system, a multitude of planning tasks has to be taken into account. Guide (2000) illustrates in his work the complexity of possible obstacles to overcome during this planning process. One of the most complex issues mentioned in his work is lot sizing for remanufacturing, i.e. the question of when to remanufacture returned products and how many items to include in each remanufacturing batch. As, in general, the entire customer demand cannot be satisfied by remanufacturing, a number of new products needs to be manufactured in addition. Incurring a setup cost for initiating a remanufacturing/manufacturing batch and holding cost for storing a returned/final product, a lot sizing problem results that needs to integrate remanufacturing and manufacturing decisions. This objective represents the main focus of this contribution.

The first attempt to find a solution to this problem has been proposed by Schrady (1967). He abstracts from a possible real-life remanufacturing system by imposing a number of assumptions to facilitate the solution finding. Most importantly, his assumption of a static product demand and return flow of products over an infinite planning horizon results in a multi-level EOQ problem setting (with EOQ being the Economic Order Quantity). In order to find a solution to this problem, Schrady separates the infinite planning horizon into equal cycles. All cycles contain the same sequence of lot sizing decisions and are repeated identically over the entire planning horizon. As commonly applied to EOQ-type lot sizing problems, the cycle needs to be determined that minimizes the total cost per time unit. Schrady recommends a cyclic solution in which $R$ equal remanufacturing lots precede a single manufacturing lot. For this kind of policy he derives closed-form expressions for the (re)manufacturing batch sizes. Further on, Schrady's proposed solution is referred to as the $(R, 1)$ policy indicating that $R$ remanufacturing batches and one manufacturing batch are set up in a cycle.

Nahmias and Rivera (1979) extend Schrady's contribution by incorporating a finite recovery rate while keeping the production rate infinitely large. In their contribution, they adjust the closed-form expressions for both lot sizes to respect their change to the model setting. Another extension to Schrady's basic model has been proposed by Richter (1996a,b). He includes the option to decide whether to dispose of returned products or not. While in the basic model remanufacturing is assumed to be always beneficial, Richter shows that this solution depends on the size of the variable cost of (re)manufacturing. Therefore, a variable disposal rate can influence the solution to this problem setting significantly when remanufacturing might not be beneficial in general. Coming back to Schrady's original problem setting, Teunter (2001) proposes another policy structure that promises better results for some parameter combinations. Teunter derives closed-form expressions for both lot sizes when one remanufacturing batch is succeeded by $M$ equal manufacturing lots. His solution will, thus, be referred to as the $(1, M)$ policy. Later on, Teunter (2004) extends in another contribution the work of Nahmias and Rivera to include finite recovery and production rates into the closed-form expressions for both the $(R, 1)$ and $(1, M)$ policies. All contributions introduced so far obtain closed-form expressions for the (re)manufacturing batch sizes under the assumption of a non-integer value for $R$ and $M$, respectively. Since $R$ and $M$ have to be integer to ensure feasibility, Minner (2002) proposes a methodology to correctly consider the issue of integrality.

In his first work, Teunter mentions two opportunities to improve the solutions proposed until then. First, he conjectures a more general $(R, M)$ policy (with $R, M > 1$ simultaneously) that can decrease the total cost incurred compared to the $(R, 1)$ and $(1, M)$ policies. This conjecture has been tested by Choi et al. (2007). They introduce a solution procedure that is able to derive the minimum total cost value for a more general $(R, M)$ solution while keeping all (re)manufacturing batches equal. In addition, a numerical experiment has been conducted to evaluate the possible improvements the more general $(R, M)$ policy offers. In their study, the $(R, M)$ policy is able to improve the currently proposed policies in about 0.2% of all instances with a maximum deviation of less than 0.5%. These findings have been, among other things, confirmed by Liu et al. (2009). Moreover, Konstantaras and Skouri (2010) extend the $(R, M)$ policy to include possible shortages. In order to do that, they adapt and facilitate the solution procedure introduced by Choi et al. As a result, their solution approach is valid for both the non-shortage and the shortage case.

Next to creating a more general $(R, M)$ policy structure, Teunter (2001) conjectures to allow for differently sized remanufacturing batches within a cycle to improve the solution even further. By using a Lagrange-multiplier approach, Minner and Lindner (2004) proved Teunter's conjecture to be true, i.e. policies containing differently sized remanufacturing batches can outperform policies with equal ones. Yet, they have not evaluated the potential gain differently sized remanufacturing batches can have. Feng and Viswanathan (2011) extend in their contribution the general $(R, M)$ policy by Choi et al. to include differently sized remanufacturing batches. Their approach proposes to split the entire $(R, M)$ cycle into two subcycles. Thereafter, an enumerative procedure tests whether the solution can be improved when the remanufacturing lot sizes are altered in both subcycles. Yet, within a subcycle all remanufacturing batch sizes remain equal. The main contribution of this work is to show that scheduling non-equal remanufacturing batches in a cycle proposes a significant cost reduction for some parameter classes. Furthermore, a more general optimization approach is introduced that allows to evaluate the solution quality of the preset policy structures.

The remainder of this work is organized as follows. After elaborating all assumptions required of the general problem setting in Section 2.1, Schrady's $(R, 1)$ policy and Teunter's $(1, M)$ policy are presented as in the original contributions in Sections 2.2 and 2.3. The only difference to their presentations is that a yield parameter $\beta$ is included in our contribution to consider the influence of an imperfect remanufacturing process. Afterwards, Section 2.4 presents the alternative formulation of the total cost function proposed by Minner (2002) to derive a closed-form expression for the integer number of remanufacturing and manufacturing batches in a cycle. Such a formulation has neither been included in Schrady's nor in Teunter's work. While Section 2.5 discusses the results of the preceding subsections in greater detail, Section 2.6 introduces a new policy structure, the $(R, 1)^g$ policy. Deviating from the formerly introduced $(R, 1)$ policy, this policy allows for differently sized remanufacturing lots in a cycle. More precisely, the amount to be remanufactured in a batch decreases geometrically throughout the cycle. This characteristic permits to fulfill the zero inventory property in both inventory levels, i.e. each remanufacturing lot remanufactures all returns in stock. Contrary, implementing an $(R, 1)$ policy with equal remanufacturing lots means that not necessarily all returns are remanufactured in a batch and a positive number can remain in stock. However, the $(R, 1)^g$ policy structure is a predefined structure like the $(R, 1)$ and $(1, M)$ policies which only allows to compare different policies. As no general optimization approach has yet been formulated in literature to evaluate the predefined policy structures properly, Section 3 provides an approach to obtain a benchmark solution by solving the underlying problem without presuming predefined structural characteristics. Thereafter, Section 4 conducts a numerical study by presenting a base case from literature and varying its parameters in a sensitivity analysis to assess the influence of each parameter on the solution quality. In this study, the simplified policy structures are compared to the benchmark solution in order to evaluate their performance. Finally, this work is concluded in Section 5 with a short summary and an outlook on future research opportunities.

# 2 Predefined policy structures for the two stage remanufacturing system

## 2.1 General model setting

Before analyzing the two stage remanufacturing system intensively, all necessary assumptions have to be stated. In general, the model setting presented subsequently concurs (with one exception) to the model setting introduced by Schrady. In it, an original equipment manufacturer (OEM) engaged in the area of remanufacturing represents the background. Figure 1 presents its general structure.

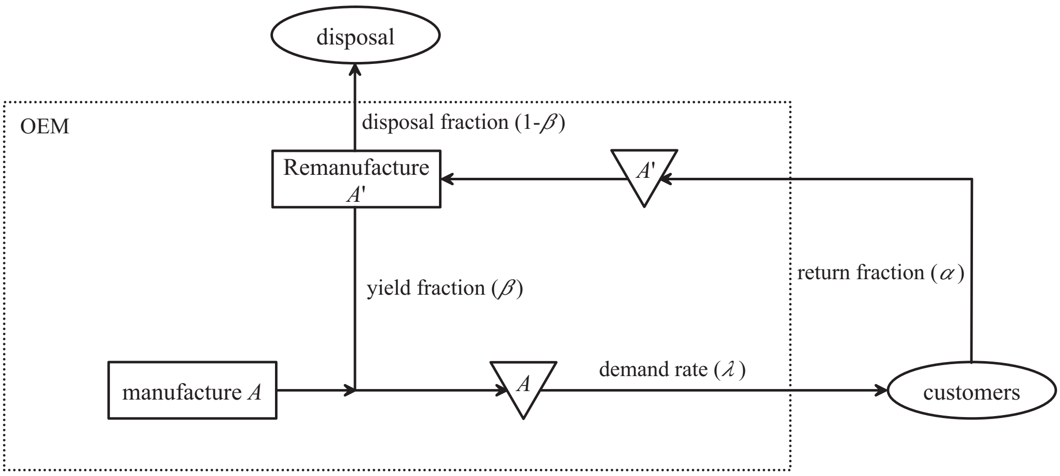


Figure 1: Inventory system in a two stage remanufacturing environment

The OEM sells one product $A$ to his customers. Demand for product $A$ is assumed to be constant and depletes the finished goods inventory continuously at a rate of $\lambda$ units per time unit. A fraction $\alpha$ of used products in the market (denoted by $A'$) returns to the manufacturer when his customers have no further use for it. Therefore, a continuous inflow of $\lambda\alpha$ returned products per time unit is observed for the used product inventory. Storing an unit of $A'$ in this inventory results in a holding cost $h_R$ per time unit. Due to different stages of wear, not all returned products can be brought to an as-good-as-new condition which is a prerequisite to resell the product. Hence, $\beta$ denotes the deterministic fraction of returned products that can be successfully reworked. Thus, $\alpha$ as well as $\beta$ must not exceed one while being non-negative. All products that cannot be remanufactured sufficiently are recycled. Recycling a returned product is assumed to be free of charge. This assumption can be imposed when the value of all materials contained in $A'$ is about the same as the value of work required to separate these materials. Setting up the mechanical rework and cleaning tools incurs a setup cost $K_R$. All successfully remanufactured products are held in a final product inventory at a cost of $h_M$ per unit per time unit. In order to secure demand for $A$ is always met, some new products have to be manufactured in addition (as $\alpha$ and $\beta$ are usually smaller than one). The relevant setup cost is denoted by $K_M$ representing the cost for initiating a manufacturing lot for product $A$. This model includes neither processing nor lead times, i.e. whenever a (re)manufacturing batch is issued it arrives instantly. Newly manufactured products are held in the same serviceables inventory as remanufactured ones. Regarding the cost of storage, both remanufactured and new products are evaluated with the same holding cost parameter $h_M$. As two levels of inventory are considered (used product and final product) the resulting system is defined as a two stage remanufacturing system.

In general, the holding costs of both inventory levels (when interpreted as opportunity cost of capital) are connected by the following condition. Since an increasing product value indicates more tied-up capital, the holding cost parameter $h_M$ must be larger than $h_R$ as the remanufacturing process provides a significant increase in value. Yet, only the fraction $\beta$ of all products returned can be sufficiently remanufactured. In other words, at an average $1/\beta$ products have to be remanufactured to obtain one sellable product. As it cannot be observed before remanufacturing whether this process is successful, the following condition for both holding cost parameters has to hold to assure validity: $h_R/\beta < h_M$. On the other hand, no condition is imposed for the process related setup costs $K_R$ and $K_M$. Contrary to these fixed cost parameters, the subsequent model omits the use of variable costs for manufacturing and remanufacturing product $A$. By assumption, remanufacturing a unit of $A$ is always less expensive than manufacturing it. Consequently, the OEM commences the remanufacturing process for all returns (whether it is successful or not) and disposes no return in advance.

Figure 2 presents the levels of inventory for the analyzed framework and depicts whether the inflows to and outflows from each level are continuous or discrete. The entire system has a continuous inflow and outflow of goods amounting to $\lambda\alpha$ and $\lambda$ units per time unit, respectively. All parameters remain constant over an infinite planning horizon which leads to an EOQ-type model (as setup and holding costs prevail). The standard single level EOQ approach recommends to replenish the inventory with a certain amount (known as the economic order quantity) whenever it is depleted. By following this simple rule over the infinite planning horizon and thereby creating identically repeated cycles, the EOQ approach minimizes the total cost per time unit. This chapter adopts the standard EOQ procedure to the more sophisticated two stage inventory problem presented above. In it, six decisions of interest have to be evaluated: the *length* of a cycle ($T$) as well as the *number* of lots scheduled therein, i.e. the number of remanufacturing ($R$) and manufacturing lots ($M$). Moreover, to define a cycle

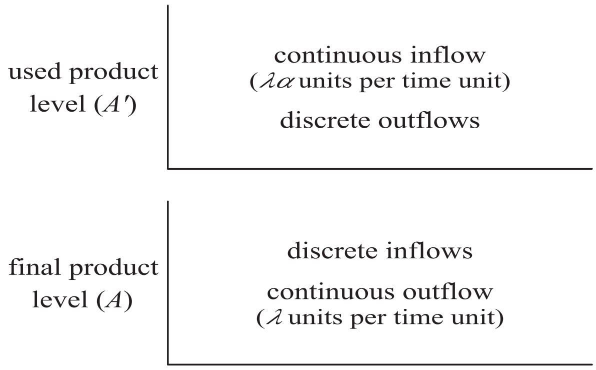


Figure 2: Two stocking points and their inflows and outflows

unambiguously, further information is required on the *sequence* of batch scheduling and on the *quantities* of individual lot sizes (denoted by $Q_R$ for remanufacturing and $Q_M$ for manufacturing lots) that need not be integer. Since all lot sizes within a cycle can be different, a complex policy structure can result. Yet, by imposing restrictions on some of these decisions, simple policy structures can be derived that facilitate finding a solution to this problem setting.

## 2.2 Schrady's $(R, 1)$ policy

The first attempt to define a simple policy structure for this problem has been undertaken by Schrady (1967). In his work, the author elaborates a set of formulae for a cyclic pattern in which one manufacturing lot is succeeded by a number of equally sized remanufacturing lots $R$. Therefore, this policy is referred to as the $(R, 1)$ policy. The simplifying assumption of having remanufacturing lots of equal size is, among other things, relaxed later on. Before doing this, the $(R, 1)$ policy with equal remanufacturing lots is presented. Figure 3 illustrates, for example, a cyclic pattern with one manufacturing and three remanufacturing lots. All lots are arranged in the way that both the used product and the final product inventories are entirely depleted at the beginning of a cycle. Thus, a cycle starts with a remanufacturing batch containing $Q_R$ returned products. Since the fraction $\beta$ can be brought to an as-good-as-new condition, $Q_R \cdot \beta$ products enter the final product inventory at the beginning of each cycle. After $\frac{Q_R \cdot \beta}{\lambda}$ time units the final product inventory is depleted and the sole manufacturing lot containing $Q_M$ final products is scheduled. Thereafter, the remaining remanufacturing lots are initiated until the end of the cycle is reached and the next, identical cycle commences. Since all remanufacturing lots are presumed to be equal, not all remanufacturable returns available in stock are remanufactured at all times. Hence, the used product level is only depleted at the beginning/end of a cycle.

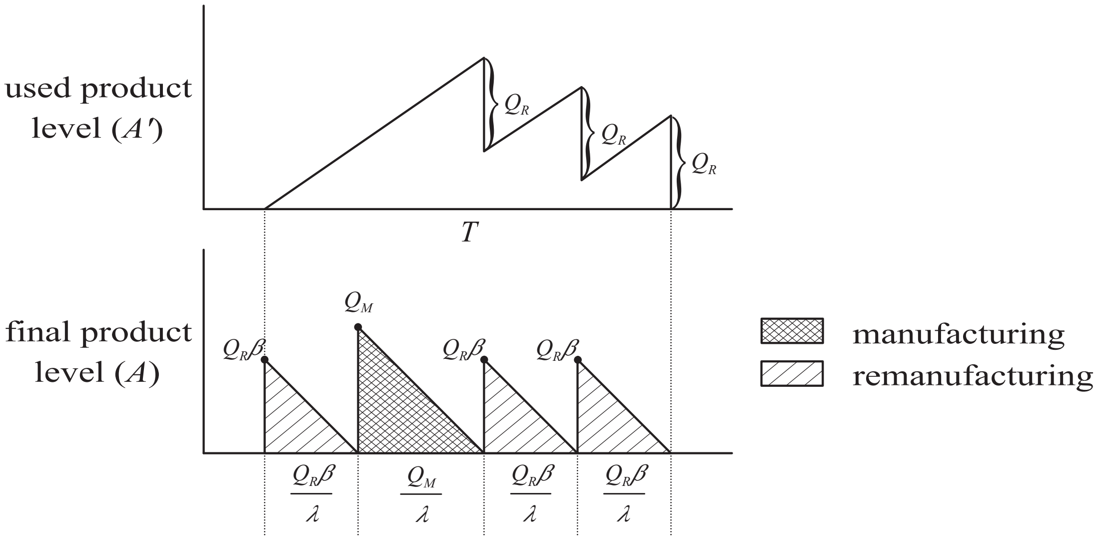


Figure 3: Used product and final product level corresponding to a (3,1) policy

Each $(R, 1)$ policy structure is unambiguously outlined by two decision variables. In his work, Schrady chooses the lot sizes $Q_R$ and $Q_M$ to evaluate the total cost of his policy structure. The remaining relevant decisions (number of remanufacturing lots $R$ and cycle length $T$) can be deduced from $Q_R$ and $Q_M$ as follows. A cyclic structure results if both inventory levels at the beginning of each cycle are equal to their respective level at the corresponding cycle's end. To ensure this, the number of returned products collected in a cycle must be equal to the amount of products remanufactured in it. Since the OEM receives $\lambda\alpha$ products per time unit and each cycle has a length of $T$ time units, $\lambda\alpha T$ products are remanufactured in $R$ identical batches of size $Q_R$, i.e.

$$R \cdot Q_R = \lambda\alpha T. \tag{1}$$

As can be derived from Figure 3, the length of a cycle $T$ is computed by

$$T(Q_R, Q_M) = \frac{R \cdot Q_R \cdot \beta + Q_M}{\lambda}. \tag{2}$$

By combining equations (1) and (2), analytical expressions can be formulated for both $R$ and $T$ that depend only on the relevant decision variables $Q_R$ and $Q_M$.

$$R(Q_R, Q_M) = \frac{\alpha \cdot Q_M}{(1-\alpha\beta) \cdot Q_R} \qquad \text{and} \qquad T(Q_M) = \frac{Q_M}{\lambda\,(1-\alpha\beta)}. \tag{3}$$

To obtain the smallest total cost of the predetermined $(R, 1)$ policy structure, the sum of a setup and a holding cost term has to be minimized. Starting with the setup cost term, the number of remanufacturing lots $R$ needs to be multiplied by $K_R$ and added to the setup cost for initiating the manufacturing batch $K_M$. The resulting value needs to be divided by the cycle length $T$ to compute the setup cost per time unit. Using equations (2) and (3), this results in[1]:

$$\frac{K_m + R \cdot K_R}{T} = \lambda \cdot \left( \frac{(1-\alpha\beta) \cdot K_M}{Q_M} + \frac{\alpha \cdot K_R}{Q_R} \right). \tag{4}$$

Regarding the holding cost term, the following analysis considers both inventories separately. The holding cost per time unit for the used product inventory can be determined by evaluating the average inventory during a cycle. In static lot sizing problems, the average inventory can be computed by dividing the maximum inventory level within a cycle $y_R^{max}$ by two. Yet, this can only be done when the inventory level of the corresponding stock is zero at the beginning and at the end of a cycle but never within. Due to the policy prerequisite of having remanufacturing lots of equal size, this is always given for an $(R, 1)$ policy structure in the used product inventory. As depicted in Figure 3, the maximum inventory in the used product stock prevails after the products fabricated in the cycle's manufacturing lot run out. At this point in time, the inventory contains all products returning to the OEM while one remanufacturing and the manufacturing lot have satisfied customer demand. As $\lambda\alpha$ products return per time unit, the holding cost for the used product stock is

$$\frac{1}{2} y_R^{max} \cdot h_R = \frac{1}{2} \cdot \alpha \cdot (Q_R \cdot \beta + Q_M) \cdot h_R. \tag{5}$$

The average holding cost in the final product inventory, on the other hand, cannot be determined by dividing the maximum inventory level during a cycle by two since it drops to zero several times in it. Generally speaking, the holding cost in a cycle is determined by multiplying the inventory during this cycle by the corresponding holding cost. The inventory during a cycle is computed by assessing the region bounded by the inventory level. For instance, to determine the holding cost of the final product level the area of the observed triangles in Figure 3 has to be evaluated. This term has to be multiplied by $h_R$ and divided by $T$ as the holding cost per time unit is required. By using equations (2) and (3), this gives[2]

$$\left( \frac{1}{2} \cdot \frac{R \cdot (Q_R \cdot \beta)^2}{\lambda} + \frac{1}{2} \cdot \frac{(Q_M)^2}{\lambda} \right) \cdot h_M \cdot \frac{1}{T} = \frac{1}{2} \left( \alpha\beta^2 \cdot Q_R + (1-\alpha\beta) \cdot Q_M \right) h_M. \tag{6}$$

After establishing the relevant setup and holding cost terms, the total cost function for Schrady's $(R, 1)$ policy depending on both lot sizes $Q_R$ and $Q_M$ is formulated by summarizing the cost components in (4), (5), and (6). Henceforth, this total cost

[1] For details, please refer to the Appendix, page 54

[2] For details, please refer to the Appendix, page 54

function is denoted by $TC_{R1}$. It is

$$TC_{R1}(Q_R, Q_M) = \lambda \cdot \left( \frac{(1-\alpha\beta) \cdot K_M}{Q_M} + \frac{\alpha \cdot K_R}{Q_R} \right) + \frac{1}{2} \cdot \alpha \cdot (Q_R \cdot \beta + Q_M) \cdot h_R + \frac{1}{2} \cdot \big( \alpha\beta^2 \cdot Q_R + (1-\alpha\beta) \cdot Q_M \big) \cdot h_M. \tag{7}$$

This total cost function (7) is jointly convex[3] in both decision variables $Q_R$ and $Q_M$, i.e. the smallest total cost can be determined by exploiting its partial derivatives. For instance, by computing the partial derivative of (7) with respect to $Q_R$, the best remanufacturing lot size $Q_R^+$ for the $(R, 1)$ policy structure is obtained. This gives

$$\frac{\partial TC_{R1}}{\partial Q_R} = -\frac{\lambda \alpha K_R}{(Q_R)^2} + \frac{1}{2} \cdot \alpha\beta \cdot (h_R + \beta \cdot h_M) = 0 \quad \text{and results in}$$

$$Q_R^+ = \sqrt{\frac{2\lambda \cdot K_R}{\beta \cdot (h_R + \beta \cdot h_M)}}. \tag{8}$$

Similarly, the best manufacturing lot size $Q_M^+$ for an $(R, 1)$ policy structure is calculated by

$$\frac{\partial TC_{R1}}{\partial Q_M} = -\frac{\lambda (1-\alpha\beta) K_M}{(Q_M)^2} + \frac{1}{2} \cdot (\alpha \cdot h_R + (1-\alpha\beta) \cdot h_M) = 0 \quad \text{and results in}$$

$$Q_M^+ = \sqrt{\frac{2\lambda \cdot (1-\alpha\beta) \cdot K_M}{\alpha \cdot h_R + (1-\alpha\beta) \cdot h_M}}. \tag{9}$$

The information about the best remanufacturing and manufacturing batch sizes can be inserted into the equations (3) to obtain the cost minimizing number of remanufacturing lots $R^+$ and cycle length $T^+$:

$$R^+ = \frac{\alpha}{1-\alpha\beta} \cdot \sqrt{\frac{(1-\alpha\beta) \cdot K_M \cdot \beta \cdot (h_R + \beta \cdot h_M)}{K_R \cdot (\alpha \cdot h_R + (1-\alpha\beta) \cdot h_M)}} \tag{10}$$

$$T^+ = \sqrt{\frac{2 \cdot K_M}{\lambda \cdot (1-\alpha\beta) \cdot (\alpha \cdot h_R + (1-\alpha\beta) \cdot h_M)}}. \tag{11}$$

When determining the optimal $(R, 1)$ policy, the number of remanufacturing lots needs to be determined as in (10). However, the number of remanufacturing lots is not necessarily integer which is a prerequisite for obtaining a feasible solution. In this case, Schrady recommends a simple rounding procedure (without exactly specifying the required rounding operations) to determine the optimal policy. In Section 2.4, an exact approach is elaborated to find a solution to this problem.

In his original work, Schrady did not consider an imperfect remanufacturing process as he assumes the yield fraction $\beta$ to be one. By introducing this fraction in the

[3]For the mathematical proof, please refer to the Appendix, page 54

above analysis, several conclusions can be drawn when comparing a situation with yield loss to a situation without it. All conclusions are supported by analyzing the first derivatives of the respective formulae with respect to $\beta$. When $\beta$ is smaller than one, a shorter cycle is recommended. As the overall number of returns decreases due to a shorter cycle, the number of remanufacturing lots per cycle decreases as well. Yet, to compensate for the yield loss and to use each remanufacturing setup efficiently, more returns are remanufactured in a setup which decreases the number of remanufacturing lots even further. Regarding the manufacturing lot size $Q_M$, no general conclusion can be drawn as the sign of the first derivative w.r.t $\beta$ depends on $K_M$ and both holding cost parameters.

Schrady's idea of creating cycles with one manufacturing lot and at least one remanufacturing lot has been discussed in literature later on. Teunter (2001) extends Schrady's work by proposing that it might be better to deviate from Schrady's $(R, 1)$ policy in some cases. His approach is introduced subsequently.

## 2.3 Teunter's $(1, M)$ policy

Contrary to Schrady's approach, Teunter proposes a preset policy structure which contains one remanufacturing and $M$ (with $M \geq 1$) manufacturing batches. This policy structure is, thus, denoted as the $(1, M)$ policy. To give an example, Figure 4 depicts a (1,2) policy. At the beginning of a cycle, the sole remanufacturing lot containing $Q_R$ returned products is initiated. Due to the imperfect remanufacturing process, only the fraction $\beta$ can be sufficiently remanufactured, i.e. $Q_R \cdot \beta$ products enter the final product stock. Since the OEM's customers request $\lambda$ products per time unit, this lot lasts for $\frac{Q_R \cdot \beta}{\lambda}$ time units. Thereafter, $M$ manufacturing lots of equal size (each comprehending $Q_M$ final products) are scheduled, each lasting for $\frac{Q_M}{\lambda}$ time units.

Similar to the $(R, 1)$ policy by Schrady, Teunter uses both lot sizes $Q_R$ and $Q_M$ to formulate the $(1, M)$ policy unambiguously, i.e. the number of manufacturing lots in a cycle $(M)$ and the cycle length $(T)$ can be deduced directly from these lot sizes. To guarantee a perfect cyclic structure, each remanufacturing lot must be of equal size. Therefore, the number of returned products in a cycle is as large as the remanufacturing lot at its beginning. Since $\lambda\alpha$ products return per time unit, the subsequent condition has to hold

$$Q_R = \lambda \alpha T. \tag{12}$$

As can be observed in Figure 4 the cycle length $T$ is computed by

$$T(Q_R, Q_M) = \frac{Q_R \cdot \beta + M \cdot Q_M}{\lambda}. \tag{13}$$

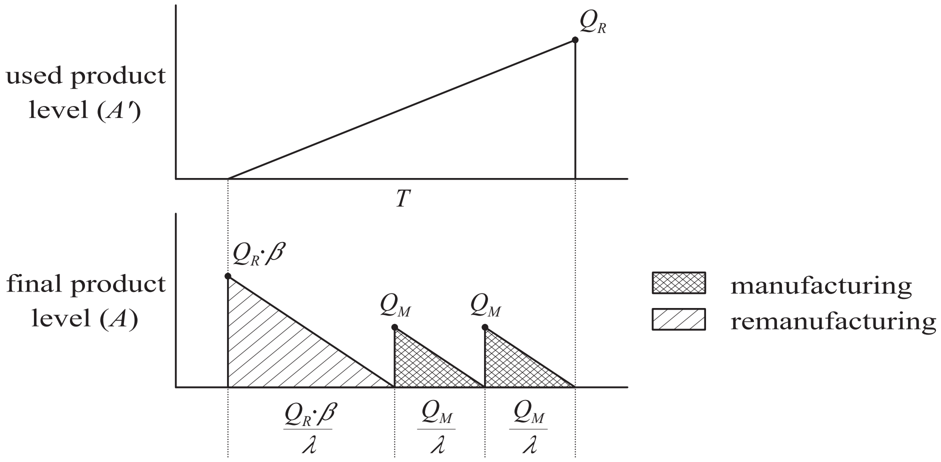


Figure 4: Used product and final product level corresponding to a (1,2) policy

Combining equations (12) and (13) provides two formulae to describe the number of manufacturing lots $M$ and the cycle length $T$ depending on $Q_R$ and $Q_M$.

$$M(Q_R, Q_M) = \frac{Q_R \cdot (1 - \alpha\beta)}{\alpha \cdot Q_M} \qquad \text{and} \qquad T(Q_R) = \frac{Q_R}{\lambda\alpha}. \tag{14}$$

When comparing conditions (3) and (14), the number of manufacturing lots $M$ for a $(1, M)$ policy is the inverse of the number of remanufacturing lots $R$ for an $(R, 1)$ policy when both are formulated depending on $Q_R$ and $Q_M$. In order to pursue the objective of minimizing the total cost per time unit, a setup and a holding cost term have to be assessed again. The former comprises the setup cost of a cycle ($M$ times the setup cost for manufacturing $K_M$ plus once the setup cost for remanufacturing $K_R$) divided by the cycle length $T$. By transformation using equations (13) and (14), the following expression is derived[4]:

$$\frac{M \cdot K_m + K_R}{T} = \lambda \cdot \left( \frac{K_M \cdot (1 - \alpha\beta)}{Q_M} + \frac{K_R \cdot \alpha}{Q_R} \right). \tag{15}$$

After formulating the setup cost, the relevant holding cost per time unit is determined. To do this, the formerly applied methodology of calculating the area bounded by both inventories during a cycle has to be used. Hence, by using equations (13) and (14), the holding cost per time unit for both inventory levels is calculated as[5]:

$$\begin{aligned} & \left[ \frac{1}{2} \cdot Q_R \cdot T \cdot h_R + \left( \frac{1}{2} \cdot \frac{(Q_R \cdot \beta)^2}{\lambda} + M \cdot \frac{1}{2} \cdot \frac{(Q_M)^2}{\lambda} \right) \cdot h_M \right] \cdot \frac{1}{T} \\ = \; & \frac{1}{2} \cdot \big( Q_R \cdot h_R + \big( \alpha\beta^2 \cdot Q_R + (1 - \alpha\beta) \cdot Q_M \big) \cdot h_M \big) . \end{aligned} \tag{16}$$

[4]For details, please refer to the Appendix, page 55

[5]For details, please refer to the Appendix, page 55

Next, the total cost function for Teunter's $(1, M)$ policy (indicated by the subindex $_{1M}$) is formulated by summarizing the cost components of (15) and (16). It is

$$
\begin{aligned}
TC_{1M}(Q_R, Q_M) =& \lambda \cdot \left( \frac{K_M \cdot (1 - \alpha\beta)}{Q_M} + \frac{K_R \cdot \alpha}{Q_R} \right) + \\
& \frac{1}{2} \cdot \left( Q_R \cdot h_R + \left( \alpha\beta^2 \cdot Q_R + (1 - \alpha\beta) \cdot Q_M \right) \cdot h_M \right). \qquad (17)
\end{aligned}
$$

Like the cost function $TC_{R1}$, the total cost function (17) is jointly convex[6] in both $Q_R$ and $Q_M$. Interestingly, the only difference between both cost functions is the evaluation of the used product's inventory which has no influence on the curvature of the total cost function but on its cost minimizing decision variables. By utilizing calculus, these variables can be computed. For instance, deriving the total cost function (17) with respect to $Q_R$ provides the optimal size of the remanufacturing lot $Q_R^+$ for a $(1, M)$ policy structure:

$$
\begin{aligned}
\frac{\partial TC_{1M}}{\partial Q_R} &= -\frac{\lambda\alpha K_R}{(Q_R)^2} + \frac{1}{2} \cdot \left( h_R + \alpha\beta^2 \cdot h_M \right) = 0 \quad \text{and, thus,} \\
Q_R^+ &= \sqrt{\frac{2\lambda\alpha \cdot K_R}{h_R + \alpha\beta^2 \cdot h_M}}. \qquad (18)
\end{aligned}
$$

Apparently, the same procedure can be applied to determine $Q_M$ as well. Thus, the optimal size of each manufacturing lot $Q_M^+$ when presuming a $(1, M)$ policy structure is derived from

$$
\begin{aligned}
\frac{\partial TC_{1M}}{\partial Q_M} &= -\frac{\lambda (1 - \alpha\beta) K_M}{(Q_M)^2} + \frac{1}{2} \cdot (1 - \alpha\beta) \cdot h_M = 0 \quad \text{which results in} \\
Q_M^+ &= \sqrt{\frac{2\lambda \cdot K_M}{h_M}}. \qquad (19)
\end{aligned}
$$

Inserting the optimal values of $Q_R^+$ and $Q_M^+$ into conditions (14) gives the cost minimizing number of manufacturing lots per cycle $M^+$ and cycle length $T^+$ for a $(1, M)$ policy structure:

$$
M^+ = \frac{(1 - \alpha\beta)}{\alpha} \cdot \sqrt{\frac{\alpha \cdot K_R \cdot h_M}{K_M \cdot (h_R + \alpha\beta^2 \cdot h_M)}} \qquad (20)
$$

$$
T^+ = \sqrt{\frac{2 \cdot K_R}{\lambda\alpha \cdot (h_R + \alpha\beta^2 \cdot h_M)}}. \qquad (21)
$$

Like for the $(R, 1)$ policy structure, the influence of including an imperfect yield $\beta$ when initiating an $(1, M)$ policy is analyzed. For instance, the cost minimizing manufacturing lot size $Q_M^+$ is not influenced at all. On the contrary, the remanufacturing lot size $Q_R^+$ increases to efficiently compensate the yield loss with respect to the setup cost.

[6] For the mathematical proof, please refer to the Appendix, page 55

Hence, the cycle length $T$ increases as more returns need to be collected. A longer cycle means that more new products are required to satisfy customer demand which results in an increasing number of manufacturing lots per cycle as the manufacturing lot size remains constant. As in the preceding subsection, these logically drawn conclusions can be derived as well by analyzing the slope of the respective cost minimizing formulae with respect to $\beta$.

After establishing the $(1, M)$ and $(R, 1)$ policy structures, it is worth mentioning that both total cost functions yield the same result in a (1,1) scenario. However, using both policies to determine a feasible solution requires both $R^+$ and $M^+$ to be integer. Considering the cost minimizing values for $R^+$ in equation (10) and $M^+$ in equation (20) depicts that this is not the case in general. While Teunter omits to discuss this issue in his contribution, Schrady mentions it briefly by proposing a rounding procedure without clearly specifying the exact rounding operation. Minner (2002) continues the discussion and elaborates an interesting result by alternatively formulating the total cost functions of both policy structures. In his contribution, both total cost functions are formulated to depend on only $R$ or $M$, respectively. By doing this, the obstacle of obtaining non-integer values for $R$ and $M$ is avoided since the total cost function depends on the sole variable that is required to be integer. The next subsection focuses on deriving his findings.

## 2.4 Alternative formulation of the $(R, 1)$ and $(1, M)$ policies

To define the total cost function of their policy structures unambiguously, Schrady and Teunter use both lot sizes $Q_R$ and $Q_M$ as their relevant decision variables. However, by inserting one of the cost minimizing lot sizes $Q_R^+$ (or alternatively $Q_M^+$) into the total cost function, the number of relevant decision variables can be reduced by one. Nevertheless, the obstacle of ensuring the number of remanufacturing (or manufacturing) lots to be integer remains to be solved. Therefore, Minner reformulates the total cost functions of both policy structures to depend on either $R$ for an $(R, 1)$ policy or $M$ for a $(1, M)$ policy structure. As two decision variables are required at the beginning, Minner chooses the cycle length $T$ to be the second one.

For the $(R, 1)$ policy, the number of remanufacturing lots per cycle can exceed one while the number of manufacturing lots is exactly equal to one. Since all remanufacturing batches are of equal size, the amount of products returning in a cycle ($\lambda\alpha T$) has to be divided by $R$ to obtain the size of each individual lot. Likewise, the amount to be manufactured in each cycle is given by the demand for the considered product that cannot be met by remanufacturing returned products, i.e. $(1 - \alpha\beta)$ of the entire demand. Therefore, the corresponding lot sizes can be reformulated (depending on $R$

and $T$) according to formulae (1) and (3) as

$$Q_R = \frac{\lambda\alpha T}{R} \quad \text{and} \quad Q_M = \lambda\,(1-\alpha\beta)\,T. \tag{22}$$

The setup cost per time unit is defined according to formula (4) which gives

$$\frac{R\cdot K_R + K_M}{T}. \tag{23}$$

Both holding cost elements can be simplified as well. Starting with the holding cost for the used product stock, the maximum inventory level in a cycle has to be evaluated. Corresponding to equation (5) this results in

$$\begin{aligned}\frac{1}{2}\cdot\alpha\cdot(Q_R\cdot\beta+Q_M)\cdot h_R =& \frac{1}{2}\cdot\alpha\cdot\left(\frac{\lambda\alpha T}{R}\cdot\beta+\lambda\cdot(1-\alpha\beta)T\right)\cdot h_R\\ =&\frac{1}{2}\lambda T\left(1+\alpha\beta\left(\frac{1}{R}-1\right)\right)\cdot\alpha h_R.\end{aligned} \tag{24}$$

In compliance with equation (6), the holding cost per time unit for the final product inventory is reformulated as

$$\begin{aligned}\frac{1}{2}\cdot\left(\alpha\beta^2\cdot Q_R+(1-\alpha\beta)\cdot Q_M\right)\cdot h_M =& \frac{1}{2}\cdot\left(\alpha\beta^2\cdot\frac{\lambda\alpha T}{R}+\lambda\cdot(1-\alpha\beta)^2\,T\right)\cdot h_M\\ =&\frac{1}{2}\lambda T\cdot\left(\frac{\alpha^2\beta^2}{R}+(1-\alpha\beta)^2\right)\cdot h_M.\end{aligned} \tag{25}$$

By adding up the setup and holding cost terms, the total cost function for the $(R,1)$ policy is established such that it depends on both $R$ and $T$:

$$TC_{R1}(R,T) = \frac{RK_R+K_M}{T}+\frac{1}{2}\lambda T\left(\left(1+\alpha\beta\left(\frac{1}{R}-1\right)\right)\alpha h_R+\left(\frac{\alpha^2\beta^2}{R}+(1-\alpha\beta)^2\right)h_M\right). \tag{26}$$

For any given value of $R$, the optimal cycle length $T$ can be computed by calculus. Thereby, the cycle length needs to be determined for which the partial derivative of the total cost function with respect to $T$ is zero. This gives

$$\frac{\partial TC_{R1}}{\partial T} = -\frac{RK_R+K_M}{T^2}+\frac{1}{2}\lambda\left(\left(1+\alpha\beta\left(\frac{1}{R}-1\right)\right)\alpha h_R+\left(\frac{\alpha^2\beta^2}{R}+(1-\alpha\beta)^2\right)h_M\right)=0$$

$$\text{and, thus,}\quad T^+_{R1}(R) = \sqrt{\frac{2\,(RK_R+K_M)}{\lambda\left(\left(1+\alpha\beta\left(\frac{1}{R}-1\right)\right)\alpha h_R+\left(\frac{\alpha^2\beta^2}{R}+(1-\alpha\beta)^2\right)h_M\right)}}. \tag{27}$$

Inserting $T^+_{R1}$ into the total cost function $TC_{R1}$ yields an expression that only depends on the number of remanufacturing lots $R$

$$TC^+_{R1}(R) = \sqrt{2\lambda(RK_R+K_M)\left(\left(1+\alpha\beta\left(\frac{1}{R}-1\right)\right)\alpha h_R+\left(\frac{\alpha^2\beta^2}{R}+(1-\alpha\beta)^2\right)h_M\right)}. \tag{28}$$

The cost minimizing number of remanufacturing lots $R$ can, thus, be computed by deriving function (28) with respect to $R$. Not surprisingly, this value matches exactly equation (10) and is therefore omitted to be presented again. Yet, the reformulation of the total cost function allows to determine the cost minimizing integer value of $R$. When analyzing function (28) in the relevant range ($R > 0$), several characteristics can be derived. First, formula (10) proves that there is only a single optimal value for $R$ minimizing the total cost function. Moreover, the total cost function approaches infinity when $R$ moves closer both to zero as well as to $+\infty$[7]. From that it follows that the local minimum determined by (10) is a global minimum for the relevant range. Exploiting these characteristics, a general procedure can be applied to determine the cost minimizing integer value $R^*$. Figure 5 depicts the optimal total cost function $TC^+_{R1}$ around its optimal non-integer value $R^+$. In it, we can observe that $R^+$ and $R^*$ lie between $\hat{R}$ and $\hat{R}+1$ which do not have to be integer but have to fulfill the condition $TC^+_{R1}(\hat{R}) = TC^+_{R1}(\hat{R}+1)$. This means the total cost function yields the same result for both values.

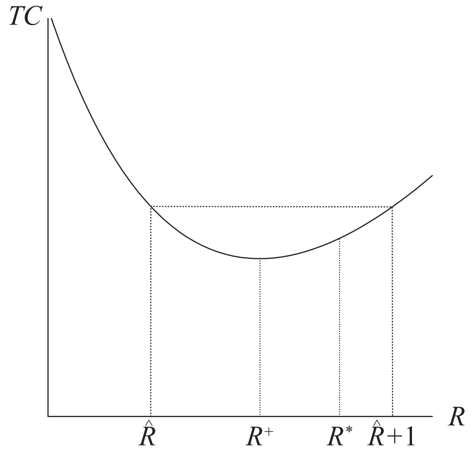


Figure 5: Total cost function $TC^+_{R1}$

There is only one integer value for $R$ between $\hat{R}$ and $\hat{R}+1$. This value must therefore be the cost minimizing integer solution $R^*$. Consequently, the value of $\hat{R}$ simply needs to be rounded up to compute $R^*$. In the case that $\hat{R}$ is an integer itself, $\hat{R}$ as well as $\hat{R}+1$ are both cost minimizing. $R^*$ is determined by[8]

$$TC^+_{R1}(\hat{R}) = TC^+_{R1}(\hat{R}+1) \quad \text{which results in}$$

$$R^* = \left\lceil -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{K_M \alpha^2 \beta \cdot (h_R + h_M \beta)}{K_R \cdot \left(\alpha\,(1-\alpha\beta)\,h_R + (1-\alpha\beta)^2\,h_M\right)}} \right\rceil . \tag{29}$$

[7]For the mathematical proof, please refer to the Appendix, page 56

[8]For details, please refer to the Appendix, page 56

Since only a positive number of remanufacturing lots is allowed, an unequivocal value for $R^*$ can be determined. Moreover, the general function $\lceil -0.5 + x \rceil$ describes the same term as if $x$ is rounded to the nearest integer. Thus, the cost minimizing integer number of remanufacturing lots for an $(R, 1)$ policy structure is computed by the following value ($\updownarrow$ indicates rounding to the nearest integer)

$$R^* = \sqrt{\frac{1}{4} + \frac{K_M \alpha^2 \beta \cdot (h_R + h_M \beta)}{K_R \cdot \left(\alpha \left(1 - \alpha\beta\right) h_R + \left(1 - \alpha\beta\right)^2 h_M\right)}} \updownarrow . \tag{30}$$

This value corresponds to the optimal value of $R^+$ determined by Schrady in equation (10) except that a quarter is added to the radicand and the resulting value is rounded to the nearest integer afterwards. The same kind of analysis can be conducted for a $(1, M)$ policy.

For the $(1, M)$ policy structure, the decision variables introduced by Teunter ($Q_R$ and $Q_M$) are replaced as well by functional expressions depending on the cycle length $T$ and the number of manufacturing lots $M$. Similar to the adaptations presented above, the (re)manufacturing batch sizes $Q_R$ and $Q_M$ are reformulated according to formulae (12) and (13) as

$$Q_R = \lambda \alpha T \qquad \text{and} \qquad Q_M = \frac{\lambda (1 - \alpha\beta) T}{M}. \tag{31}$$

By implementing equations (31), the reformulation of the setup cost per time unit is facilitated. Analogous to equation (15), this results in

$$\frac{K_R + M \cdot K_M}{T}. \tag{32}$$

To obtain the holding cost per time unit for a $(1, M)$ policy in the alternative formulation, formulae (31) are used to adapt equation (16):

$$\begin{aligned} & \left[\frac{1}{2} \cdot Q_R \cdot T \cdot h_R + \left(\frac{1}{2} \cdot \frac{\left(Q_R \cdot \beta\right)^2}{\lambda} + M \cdot \frac{1}{2} \cdot \frac{\left(Q_M\right)^2}{\lambda}\right) \cdot h_M\right] \cdot \frac{1}{T} \\ = \; & \frac{1}{2} \lambda T \cdot \left(\alpha h_R + \left(\alpha^2 \beta^2 + \frac{\left(1 - \alpha\beta\right)^2}{M}\right) \cdot h_M\right). \end{aligned} \tag{33}$$

The total cost per time unit results from the sum of the setup cost (32) and holding cost (33) per time unit. Hence,

$$TC_{1M}(M, T) = \frac{K_R + M \cdot K_M}{T} + \frac{1}{2} \lambda T \cdot \left(\alpha h_R + \left(\alpha^2 \beta^2 + \frac{\left(1 - \alpha\beta\right)^2}{M}\right) \cdot h_M\right). \tag{34}$$

In analogy to the procedure for the $(R, 1)$ policy structure, the optimal cycle length $T_{1M}^+$ and the corresponding minimizing total cost function $TC_{1M}^+$ depending only on the number of manufacturing lots $M$ can be determined.

$$T_{1M}^{+}(M) = \sqrt{\frac{2 \cdot (K_R + M \cdot K_M)}{\lambda \cdot \left(\alpha h_R + \left(\alpha^2\beta^2 + \frac{(1-\alpha\beta)^2}{M}\right) \cdot h_M\right)}}$$

$$TC_{1M}^{+}(M) = \sqrt{2\lambda \cdot (K_R + M \cdot K_M) \cdot \left(\alpha h_R + \left(\alpha^2\beta^2 + \frac{(1-\alpha\beta)^2}{M}\right) \cdot h_M\right)}. \tag{35}$$

The total cost function (35) reveals the same characteristics as the total cost function for an $(R, 1)$ policy structure, i.e. it has a single minimum and approaches infinity for $M \to 0$ and $M \to \infty$[9]. Therefore, the same methodology can be applied as for the $(R, 1)$ policy. Let $\hat{M}$ denote the value of $M$ that needs to be rounded up to obtain the cost minimizing integer number of manufacturing batches in a cycle. We find[10]

$$TC_{1M}^{+}(\hat{M}) = TC_{1M}^{+}(\hat{M}+1) \quad \text{and, thus,}$$

$$M^* = \left\lceil -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{K_R \cdot (1-\alpha\beta)^2 \cdot h_M}{K_M \cdot (\alpha h_R + \alpha^2\beta^2 h_M)}} \right\rceil. \tag{36}$$

When comparing the results of the $(R, 1)$ policy with the results of the $(1, M)$ policy in equation (36), the outcome is quite similar. Thus, the cost minimizing integer number of manufacturing lots in a cycle is computed by adding a quarter to the radicand of Teunter's solution in equation (20) and rounding the resulting value to the nearest integer. This means

$$M^* = \sqrt{\frac{1}{4} + \frac{K_R \cdot (1-\alpha\beta)^2 \cdot h_M}{K_M \cdot (\alpha h_R + \alpha^2\beta^2 h_M)}} \updownarrow. \tag{37}$$

Deriving closed-form expressions for $R^*$ and $M^*$ has been one of the main results of Minner's contribution. However, both values can never be smaller than 1 (as he presumed) since the radicand is at least 0.25, i.e. its square root is at least 0.5. As this value has to be rounded to the nearest integer afterwards, the optimal values for $R^*$ and $M^*$ are always at least equal to 1.

Concluding, the optimal parameter $R^*$ for an $(R, 1)$ policy can be determined by equation (30). Likewise, $M^*$ can be computed using (37) to get the optimal $(1, M)$ policy. For a given set of parameters, the resulting optimal total cost functions $TC_{R1}^{+}(R^*)$ and $TC_{1M}^{+}(M^*)$ would have to be compared to find the better solution. The next subsection proves that this is not necessary as $R^*$ and $M^*$ cannot exceed a value of one simultaneously when restricting oneself to the $(R, 1)$ and $(1, M)$ policies.

[9]We omit to present the mathematical proof as it is similar to the proof for the $(R, 1)$ policy
[10]For details, please refer to the Appendix, page 57

## 2.5 Comparison of the optimal values for $R^*$ and $M^*$

At the beginning of this subsection, a small example illustrates the implications when $R^*$ and $M^*$ would not be larger than one at the same time. Assume that by applying formula (30) to an exemplary set of parameters, two remanufacturing lots should be initiated when considering an $(R, 1)$ policy structure, i.e. $R^* = 2$. Consequently, $M^*$ would have to be one as the conjecture to be proven states that both $R^*$ and $M^*$ cannot be larger than one simultaneously. Therefore, the best $(1, M)$ policy structure would be a (1,1) policy. This policy is, however, outperformed by the (2,1) policy structure since a (1,1) policy is a possible $(R, 1)$ policy structure as well. Thus, the decision maker would simply need to calculate the optimal values for $R^*$ and $M^*$ using formulae (30) and (37) to obtain the best policy parameters for the predetermined policy structures. Hence, a comparison of both minimal cost values of the $(R, 1)$ and $(1, M)$ policies could be omitted.

In order to prove the above conjecture, two inequalities would have to hold simultaneously. At first, $R^*$ determined by formula (30) has to be larger than 1.5 since its value rounded to the nearest integer is consequently greater or equal to two. This gives

$$\sqrt{\frac{1}{4} + \frac{K_M \alpha^2 \beta \cdot (h_R + h_M \beta)}{K_R \cdot \left(\alpha (1 - \alpha\beta) h_R + (1 - \alpha\beta)^2 h_M\right)}} \geq 1.5 \quad \text{and, thus,}$$
$$\frac{1}{2} K_M \alpha^2 \beta \cdot (h_R + h_M \beta) - K_R \cdot \alpha (1 - \alpha\beta) h_R \geq K_R \cdot (1 - \alpha\beta)^2 h_M. \tag{38}$$

If condition (38) is fulfilled, more than one remanufacturing lot should be initiated in a cycle ($R^* \geq 2$) when applying the $(R, 1)$ policy. In this case, the number of manufacturing lots is set to one due its predefined policy structure.

Next, the same analysis is put forth for the $(1, M)$ policy by accordingly evaluating condition (37). We find

$$\sqrt{\frac{1}{4} + \frac{K_R \cdot (1 - \alpha\beta)^2 \cdot h_M}{K_M \cdot (\alpha h_R + \alpha^2 \beta^2 h_M)}} \geq 1.5 \quad \text{which results in}$$
$$2 K_M \cdot \left(\alpha h_R + \alpha^2 \beta^2 h_M\right) \leq K_R \cdot (1 - \alpha\beta)^2 \cdot h_M. \tag{39}$$

Condition (39) needs to hold if more than one manufacturing lot should be scheduled in a cycle ($M^* \geq 2$) when applying the $(1, M)$ policy. Without loss of generality, the non-strict inequalities are replaced by strict inequalities. If two strict inequalities have to hold at the same time, it is possible to subtract them and analyze the validity of the resulting inequality. This gives

$$\frac{1}{2} K_M \alpha^2 \beta \cdot (h_R + h_M \beta) - K_R \cdot \alpha (1 - \alpha\beta) h_R - 2 K_M \cdot \left(\alpha h_R + \alpha^2 \beta^2 h_M\right) > 0$$
$$K_M \alpha \cdot \left(\left(\frac{1}{2}\alpha\beta - 2\right) h_R - \frac{3}{2}\alpha\beta^2 h_M\right) - K_R \cdot \alpha (1 - \alpha\beta) h_R > 0. \tag{40}$$

As all parameters are positive and $\alpha\beta$ cannot exceed one, the term on the left hand side of inequality (40) is always negative. Hence, this inequality is never satisfied, i.e. conditions (38) and (39) never hold simultaneously. This means, $R^*$ and $M^*$ can never be larger than one at the same time when restricting oneself to the preset $(R, 1)$ and $(1, M)$ policies. However, this result is only valid for these two policy structures. Choi et al. (2007) have shown in their work, for instance, that a more general $(R, M)$ policy with both $R$ and $M$ larger than one can reduce the resulting total cost.

After introducing the $(R, 1)$ and $(1, M)$ policy structures it has to be mentioned that their solution quality is hardly discussed in literature. Minner and Lindner (2004), for instance, elaborate in their contribution that it might not be optimal to choose remanufacturing lots of equal size in a cycle. This topic is discussed more intensively in the next subsection. There, a third preset policy structure is introduced which allows for different remanufacturing batches in a cycle.

## 2.6 The $(R, 1)^g$ policy

When non-equal remanufacturing lots are allowed in a cycle, a multitude of alternative policy structures can be formulated. In their article, Minner and Lindner apply a Lagrange-multiplier approach to investigate the optimality of having remanufacturing lots of equal size in an $(R, 1)$ policy. As a result, they identify three cases which have in common that differently sized remanufacturing batches are initiated within each cycle. The first case is to have $R - 1$ remanufacturing lots of equal size which are succeeded by a smaller last one. The second case comprises that all remanufacturing lots in a cycle decrease geometrically. Finally, the third case incorporates a mix of the first two, i.e. a number of equally sized remanufacturing batches is followed by a number of geometrically decreasing ones.

Minner and Lindner restrict their analysis to identifying these three cases. However, the subsequent analysis focuses on the second case as this case is the only one having a special characteristic. When scheduling geometrically decreasing remanufacturing lots in a cycle, each lot remanufactures all currently available returns. Such a schedule (that fulfills the zero inventory property) is easy to apply and can neither be implemented for a regular $(R, 1)$ policy with equal remanufacturing lots (see, for instance, Figure 3) nor for the remaining two cases identified by Minner and Lindner. Figure 6 presents the used product and final product inventories in a cycle when two geometrically decreasing remanufacturing lots are initiated. To differ this policy from a regular $(2, 1)$ policy, it is denoted by $(2, 1)^g$ to indicate the geometrically decreasing remanufacturing batches.

By definition, the largest remanufacturing batch in a cycle is denoted by $Q_{R,1}$. It comprehends all products collected by the OEM while the smallest remanufacturing lot

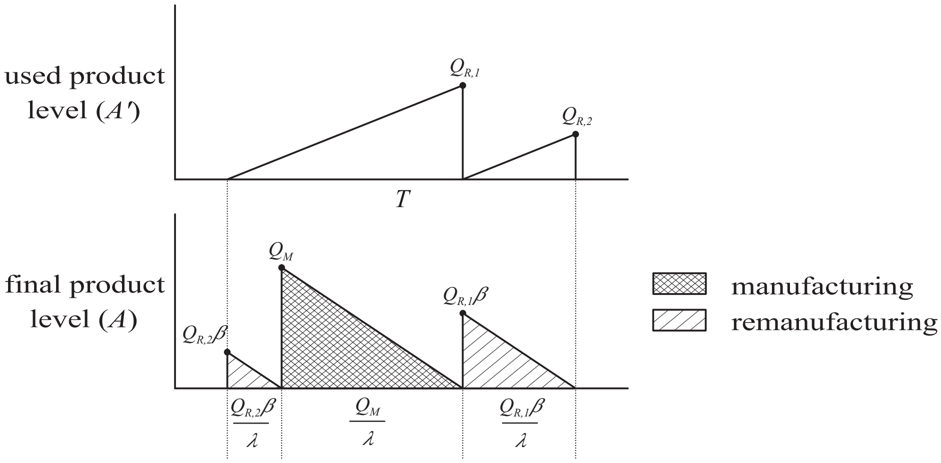


Figure 6: Used product and final product level corresponding to a $(2,1)^g$ policy

(denoted by $Q_{R,R}$) and the manufacturing lot satisfy customer demand. Beginning with the largest remanufacturing batch, each subsequently scheduled lot remanufactures only $\alpha\beta$ of its predecessor's lot size. This fact is illustrated using $Q_{R,1}$ and $Q_{R,2}$ from Figure 6. $Q_{R,1}$ satisfies demand for exactly $Q_{R,1} \cdot \beta/\lambda$ time units. During that time interval, the collection of returns for the second remanufacturing lot $Q_{R,2}$ takes place. Over a time span of $Q_{R,1} \cdot \beta/\lambda$ time units $\lambda\alpha$ products are accumulated per time unit. Therefore, as all collected products have to be remanufactured, $Q_{R,2}$ comprises $\lambda\alpha \cdot Q_{R,1} \cdot \beta/\lambda = \alpha\beta \cdot Q_{R,1}$ units. To implement geometrically decreasing lots, two conditions have to be respected. First, as shown previously each remanufacturing lot (except $Q_{R,1}$) remanufactures $\alpha\beta$ of its predecessor's batch size. Second, all returned products must be remanufactured during a cycle, i.e. $\sum_{i=1}^{R} Q_{R,i} = \lambda\alpha T$. Respecting these two conditions, an expression can be derived which describes the size of each remanufacturing lot for the $(R,1)^g$ policy. Hence[11],

$$Q_{R,i} = \frac{\lambda\alpha^i\beta^{i-1}T \cdot (1-\alpha\beta)}{1-\alpha^R\beta^R} \qquad \forall i = 1,..,R. \tag{41}$$

After formulating the amount to be remanufactured in each lot, the total cost function is established. The setup cost per time unit can be computed similar to an $(R,1)$ policy structure by the following formula:

$$\frac{R \cdot K_R + K_M}{T}. \tag{42}$$

Due to their complexity, the holding cost terms for both inventories are analyzed separately. Starting with the used product inventory and using equations (41), the holding

[11]For details, please refer to the Appendix, page 58

cost per time unit for this inventory is formulated as[12]

$$\frac{1}{2}\lambda\alpha T h_R \cdot \left(\frac{1-\alpha\beta}{1+\alpha\beta} \cdot \frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}\right). \tag{43}$$

Similarly, the holding cost for the final product inventory is computed. The size of the cycle's manufacturing lot corresponds to its size for a regular $(R,1)$ policy, i.e. $Q_M = \lambda(1-\alpha\beta)T$. Therefore, we obtain[13]

$$\frac{1}{2}\lambda T h_M \left(\alpha^2\beta^2 \cdot \frac{1-\alpha\beta}{1+\alpha\beta} \cdot \frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R} + (1-\alpha\beta)^2\right). \tag{44}$$

The total cost per time unit for a $(R,1)^g$ policy is then calculated by summing up the cost components (42), (43), and (44). Until now, the total cost function depends on both the number of remanufacturing batches $R$ and the cycle length $T$. It is

$$TC_{R1^g}(R,T) = \frac{RK_R+K_M}{T} + \frac{1}{2}\lambda T\left((\alpha h_R+\alpha^2\beta^2 h_M)\cdot\left(\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}\right)+h_M(1-\alpha\beta)^2\right). \tag{45}$$

Similar to the approaches presented above, the total cost can be adapted to depend only on the number of remanufacturing lots in a cycle $R$. For this, the cost minimizing cycle length $T^+_{R1^g}$ needs to be computed and inserted into the total cost function. This gives

$$T^+_{R1^g}(R) = \sqrt{\frac{2\cdot(RK_R+K_M)}{\lambda\left((\alpha h_R+\alpha^2\beta^2 h_M)\cdot\left(\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}\right)+h_M(1-\alpha\beta)^2\right)}} \quad \text{and, thus,}$$

$$TC^+_{R1^g}(R) = \sqrt{2\lambda(RK_R+K_M)\left((\alpha h_R+\alpha^2\beta^2 h_M)\left(\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}\right)+h_M(1-\alpha\beta)^2\right)}. \tag{46}$$

Although depending only on $R$, no closed-form expression exists to calculate the cost minimizing value of $R$ since it can be found both in the base and exponent of equation (46). To obtain a greater insight into the total cost function's behavior, a large set of different problem instances has been created. Without exception, the cost function always had only one cost minimum. Based on this observation, a simple local search method is recommended to determine the optimal value for $R$.

After formulating the total cost function for an $(R,1)^g$ policy structure, two interesting insights can be derived. At first, the condition required for a $(2,1)^g$ policy to outperform a (1,1) policy is depicted. When setting $R$ equal to one, the total cost function of the $(R,1)^g$ policy matches exactly the total cost function of the $(R,1)$ policy.

[12]For details, please refer to the Appendix, page 58

[13]For details, please refer to the Appendix, page 59

Therefore, after replacing $\frac{1-\alpha\beta}{1+\alpha\beta} \cdot \frac{1+\alpha^2\beta^2}{1-\alpha^2\beta^2}$ by $V$, the following condition results[14]

$$\text{from} \quad TC^{+}_{R1^g}(1) - TC^{+}_{R1^g}(2) > 0:$$
$$K_R\left(\alpha h_R + \alpha^2\beta^2 h_M\right)(1-2V) - K_R h_M (1-\alpha\beta)^2 + K_M\left(\alpha h_R + \alpha^2\beta^2 h_M\right)(1-V) > 0. \tag{47}$$

If condition (47) holds, the number of remanufacturing lots in an $(R,1)^g$ policy should be larger than one. The preceding subsection 2.5 has proven that $R$ and $M$ can never be larger than one simultaneously when restricting to the $(R,1)$ and $(1,M)$ policy structures. An interesting question arises whether the same can be proven for the $(R,1)^g$ and $(1,M)$ policies. For this to be true, conditions (39) and (47) must not hold simultaneously. As before, this can be examined by subtracting these inequalities and analyzing the resulting inequality. We find

$$K_R\left(\alpha h_R + \alpha^2\beta^2 h_M\right)(1-2V) - K_M\left(\alpha h_R + \alpha^2\beta^2 h_M\right)(1+V) > 0. \tag{48}$$

The resulting inequality (48) can never be fulfilled as long as $V$ is always larger than 0.5 since then both terms on the left hand side of (48) are strictly negative. The following calculations prove that this is the case[15]:

$$\frac{1-\alpha\beta}{1+\alpha\beta} \cdot \frac{1+\alpha^2\beta^2}{1-\alpha^2\beta^2} > \frac{1}{2} \quad \text{which results in}$$
$$(1-\alpha\beta)^3 > 0. \tag{49}$$

Since $\alpha\beta$ always lies between zero and one, $(1-\alpha\beta)^3$ is always positive. Inequality (48), thus, is never fulfilled. Therefore, if the local search applied to equation (46) computes a cost minimizing value of $R$ larger than one, the best possible $(1,M)$ policy would be a (1,1) structure which is in this case outperformed by the best $(R,1)^g$ policy.

The $(R,1)^g$ policy structure cannot only be compared to the $(1,M)$ policy but also to the $(R,1)$ policy with equal remanufacturing lots. As this cannot be done in general, a condition is derived at which a $(2,1)^g$ policy outperforms a $(2,1)$ policy. To do so, the total cost functions (28) and (46) have to be subtracted for $R$=2. From[16]

$$TC^{+}_{R1}(2) - TC^{+}_{R1^g}(2) > 0 \quad \text{we find}$$
$$\frac{h_M}{h_R} < \frac{3+\alpha\beta}{\beta(1-\alpha\beta)}. \tag{50}$$

Four parameters determine whether a $(2,1)^g$ policy with geometrically decreasing remanufacturing lots outperforms a (2,1) policy with equal remanufacturing batches: both holding cost parameters and their relation as well as the return and yield fractions

[14]For details, please refer to the Appendix, page 59
[15]For details, please refer to the Appendix, page 59
[16]For details, please refer to the Appendix, page 60

$\alpha$ and $\beta$. The relation between $h_M$ and $h_R$ influences the result of the analysis substantially. It says that for large values of $h_M$ compared to $h_R$, equal remanufacturing batches are preferred. Otherwise, geometrically decreasing remanufacturing batches should be initiated when the ratio between $h_M$ and $h_R$ is comparably small. The exact value is depicted for $R = 2$ in (50). Interestingly, the value of the right hand side of (50) approaches infinity when $\alpha$ and $\beta$ move closer to either zero or one. In these settings, geometrically decreasing remanufacturing lots are mostly preferred over lots of equal size. In the following, the right hand side of inequality (50) is analyzed in greater detail. By deriving it with respect to $\alpha$ the impact of the return fraction is evaluated. It gives

$$\frac{\partial \frac{3+\alpha\beta}{\beta(1-\alpha\beta)}}{\partial \alpha} = \frac{4}{(1-\alpha\beta)^2}. \tag{51}$$

Since this term is strictly positive, $\frac{3+\alpha\beta}{\beta(1-\alpha\beta)}$ increases if $\alpha$ becomes larger. Hence, a larger return fraction benefits the $(2,1)^g$ policy over the (2,1) structure. The same kind of analysis is also conducted for the yield parameter $\beta$.

$$\frac{\partial \frac{3+\alpha\beta}{\beta(1-\alpha\beta)}}{\partial \beta} = \frac{-3+6\alpha\beta+\alpha^2\beta^2}{\beta^2(1-\alpha\beta)^2}. \tag{52}$$

Contrary to the analysis of the return fraction, an ambiguous result is derived for $\beta$. While the denominator of (52) is positive, the numerator's sign depends on the value of $\alpha$. If the return fraction $\alpha$ is smaller than $-3+\sqrt{12}$ (around 46.4%), the right hand side of (50) decreases continuously as larger $\beta$ becomes, i.e. a (2,1) policy becomes more attractive as $\beta$ rises. If, on the other hand, $\alpha$ is larger than $-3+\sqrt{12}$, an increasing $\beta$ lets the value of $\frac{3+\alpha\beta}{\beta(1-\alpha\beta)}$ fall until it reaches a minimum but begins to rise thereafter until $\beta$ approaches one.

The results regarding $\alpha$ and $\beta$ are supported by logical conclusions. Figure 7 confronts the result of a (2,1) with a $(2,1)^g$ policy. Both policies represent trade-off solutions when regarding them from an efficiency point of view. Implementing a (2,1) policy, for instance, is the perfect (because least costly) policy with one manufacturing and two remanufacturing batches for the final product level. Yet, the used product level needs to deviate from a good solution by scheduling a remanufacturing lot that does not remanufacture all returned products on hand. Contrary, a $(2,1)^g$ policy accepts a worse solution in the final product stock by allowing to remanufacture in differently sized lots. By doing this, an efficient remanufacturing process from the used product inventory's point of view is obtained since no return remains in stock after initiating a remanufacturing batch. Hence, if holding returned products in the used product stock is relatively expensive compared to holding finished products in the final product inventory, it becomes more interesting to find an efficient solution for the used

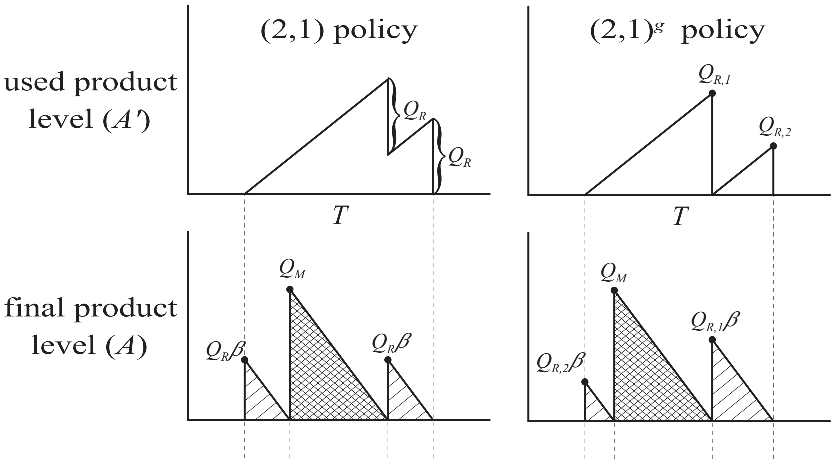


Figure 7: Comparison of a (2,1) policy to a $(2,1)^g$ policy

product level (geometrically decreasing remanufacturing lots). On the other hand, if holding finished products is very costly compared to holding returned products, the best solution for the final product level is chosen (equal remanufacturing lots).

These conclusions can be used to analyze the results with respect to $\alpha$ and $\beta$. When the fraction of returned products per time unit rises, more returns need to be held and, thus, the importance of the used product stock increases. Hence, a $(2,1)^g$ policy becomes more attractive as it focuses on an efficient solution for this inventory level. If the quality parameter $\beta$ rises, the effects cannot be seen as clearly. An increase in $\beta$ leads, ceteris paribus, to less manufactured but more remanufactured products to satisfy customer demand. Since this only shifts the batches of the final product level but has no direct influence on the used product level, a logical conclusion cannot be drawn in general.

As mentioned above, comparing a (2,1) to a $(2,1)^g$ policy structure can only give some structural insights since these results cannot be generalized. Yet, similar results as in (50) can be derived when comparing other policy structures. In Table 1, the conditions for an $(R,1)^g$ policy to dominate an $(R,1)$ policy are presented (for $R \leq 5$).

Unfortunately, no bounds can be determined on the maximum error of applying an $(R,1)$ instead of an $(R,1)^g$ policy. To illustrate the complexity of this situation, a small example is presented. For an exemplary parameter set, a policy with one manufacturing and two equal remanufacturing lots is the best option considering all $(R,1)$ and $(1,M)$ policies. For the same parameter set, three geometrically decreasing remanufacturing lots are the best solution of all possible $(R,1)^g$ policies, i.e. a $(2,1)$ policy would have to be compared to a $(3,1)^g$ policy for this parameter set. As the interdependence of all policies has to be respected, i.e. general conditions would have to be derived describing which preset policy is the best for each parameter combina-

Table 1: Comparison of policies with and without remanufacturing lots of equal size

| | Condition that needs to hold when $TC^{+}_{R1}(R) - TC^{+}_{R1^g}(R) > 0$ |
|---|---|
| For $R = 2$ | $\frac{h_M}{h_R} < \frac{3 + \alpha\beta}{\beta\,(1 - \alpha\beta)}$ |
| For $R = 3$ | $\frac{h_M}{h_R} < \frac{2 + \alpha\beta}{\beta\,(1 - \alpha\beta)}$ |
| For $R = 4$ | $\frac{h_M}{h_R} < \frac{3\alpha^3\beta^3 + 9\alpha^2\beta^2 + 7\alpha\beta + 5}{\beta\,(-3\alpha^3\beta^3 - \alpha^2\beta^2 + \alpha\beta + 3)}$ |
| For $R = 5$ | $\frac{h_M}{h_R} < \frac{2\alpha^3\beta^3 + 4\alpha^2\beta^2 + \alpha\beta + 3}{\beta\,(-2\alpha^3\beta^3 + \alpha^2\beta^2 - \alpha\beta + 2)}$ |

tion, a closed-form expression on the percentage gain cannot be formulated. Therefore, a small numerical study is conducted in Section 4 to evaluate the $(R, 1)^g$ policy with respect to the $(R, 1)$ structure. In this study, a number of instances revealed a performance gain of more than 5 % when initiating geometrically decreasing instead of equal remanufacturing batches. However, no contribution has yet established a methodology to evaluate the performance of the introduced policy structures in general. Hence, the upcoming Section 3 establishes a benchmark solution that determines for a given $R$ and $M$ the optimal solution without imposing additional constraints on the lot sizes.

# 3 Establishing a benchmark solution

In order to define a policy structure unambiguously, six decisions need to be determined: the cycle length ($T$), the number of (re)manufacturing batches ($R$ and $M$), the sequence of batch scheduling, and the corresponding (re)manufacturing batch sizes ($Q_R$ and $Q_M$). When establishing a preset policy structure, a number of decisions is fixed in advance. For the $(R, 1)^g$ policy, for instance, the number of manufacturing lots per cycle is fixed to one. Furthermore, the batch sequence in a cycle is predefined and the remanufacturing lot sizes are geometrically decreasing. The exact size of the (re)manufacturing batches depends, however, on the not yet known cycle length and the number of remanufacturing lots $R$. Therefore, by fixing the number of manufacturing lots and assuming a characteristic pattern for all remanufacturing batch sizes, the best $(R, 1)^g$ policy is obtained. Likewise, the best $(R, 1)$ and $(1, M)$ policy can be determined. The subsequently introduced optimization approach deviates from this procedure as it fixes next to the number (re)manufacturing lots ($R$ and $M$) also the cycle length. By computing the optimal solutions for a possible set of $(R, M)$ combinations, a benchmark solution can be found.

In general, the total cost of a cycle consists of its total setup cost ($SC$) that is added to the corresponding total holding cost ($HC$). As the total cost per time unit ($TC$) represents the objective of optimization, the sum of both costs has to be divided by the cycle length $T$:

$$TC = \frac{SC + HC}{T} \tag{53}$$

While the setup cost $SC$ depends on the number of remanufacturing and manufacturing batches in a cycle but not on the cycle length itself, the holding cost $HC$ depends on the cycle length, i.e. $HC = HC(T)$. Obtaining the benchmark solution to this problem exploits the dependency of $HC$ with respect to $T$. First, by fixing the number of $R$ and $M$ in a cycle, the setup cost value is also fixed. Therefore, only the size of the holding cost per cycle needs to be minimized to determine the optimal solution. Interestingly, the holding cost per cycle depends quadratically on the cycle length. This can be explained by the fact that the relative scheduling of remanufacturing and manufacturing batches in a cycle (e.g. remanufacture 20% of all returns after 60% of the cycle has passed) does not depend on its overall length. For instance, if $T$ is doubled all batch sizes are doubled, too. Hence, the time to collect the appropriate returns doubles as well as the time a (re)manufacturing lot is able to satisfy customer demand. Thus, $HC$ is going to be four times its initial value if $T$ is doubled. By defining $HC_1$ as the holding cost for a cycle length of one time unit, the condition $HC(T) = HC_1 \cdot T^2$ can be established. After inserting this condition into formula (53) the optimal cycle length and total cost can be determined by:

$$\begin{aligned} TC &= \frac{SC + HC_1 \cdot T^2}{T} = \frac{SC}{T} + HC_1 \cdot T \\ \Rightarrow T^* &= \sqrt{\frac{SC}{HC_1}} \\ \Rightarrow TC^* &= 2 \cdot \sqrt{SC \cdot HC_1} \end{aligned} \tag{54}$$

By fixing the cycle length to one time unit, the optimal batch sequence and the corresponding (re)manufacturing batch sizes can be determined as long as $R$ and $M$ are given. The most interesting aspect of this approach is that no direct relation between the (re)manufacturing batches of a cycle are imposed. In order to calculate the optimal solution for any $(R, M)$ combination, the problem is solved sequentially in two steps. These are:

*Step 1*: For a given $(R, M)$ combination, minimize $HC_1$ w.r.t. the lot sequence and (re)manufacturing lot sizes.

*Step 2:* Compute the optimal total cost and cycle length for $HC_1^*$ using formula (54).

To obtain the optimal solution for $HC_1$, the concept of subcycle-oriented optimization is employed. In this concept, the whole cycle is separated into $R$ subcycles (denoted by $s$) in which the following presumptions are required to hold. At the beginning of each subcycle the sole remanufacturing lot is initiated. It contains exactly $Q_{R,s}$ items that are remanufactured at once. If the number of remanufactured components is not sufficient to satisfy the subcycle's demand, a number of components (denoted by $\Theta_{M,s}$) has to be manufactured in $\nu_{M,s}$ equal manufacturing lots. All manufacturing lots in a subcycle should be of equal size since deviating from equal manufacturing lots in a subcycle would increase the holding cost incurred. This is shown in Appendix B of Schulz and Ferretti (2008). The individual lot size of a manufacturing lot $Q_{M,s}$ is therefore determined by $\Theta_{M,s}/\nu_{M,s}$. However, it is possible that no new component is fabricated in a subcycle, i.e. $\Theta_{M,s} = 0$. To summarize, each subcycle contains exactly one remanufacturing lot and zero, one, or more manufacturing lots.

To determine the optimal cycle when $R$ and $M$ are given, no further assumptions regarding the (re)manufacturing batches are imposed. This includes the option to have used products left in stock at the end of a subcycle as depicted in Figure 3. By including this possibility, not all products available in stock have to be remanufactured at the end of a subcycle. In the following model, $V_s$ denotes the used product inventory level at the end of subcycle $s$. On the other hand, the final product level has to be depleted at the end of each subcycle. Due to the flexibility in timing and sizing the (re)manufacturing batches, initiating one of these batches before the final product level is empty would increase the holding cost incurred since holding final products is more expensive than holding returns. Figure 8 presents a possible solution when the subcycle-oriented optimization approach is applied to a policy structure with three remanufacturing and two manufacturing lots. Without loss of generality, both inventories are set to zero at the beginning/end of a cycle, i.e. $V_R = 0$.

For a given policy structure, the minimal holding cost for a cycle length of one time unit $HC_1^*$ can be determined by the following optimization approach:

$$\begin{aligned} \min HC_1 = \quad & \frac{1}{2\lambda} \cdot \left( \frac{h_R}{\alpha} \cdot \left[ (Q_{R,1} + V_1)^2 + \sum_{s=2}^{R} \left[ (Q_{R,s} + V_s)^2 - (V_{s-1})^2 \right] \right] + \right. \\ & \left. h_M \cdot \sum_{s=1}^{R} \left[ (Q_{R,s} \cdot \beta)^2 + \frac{(\Theta_{M,s})^2}{\nu_{M,s}} \right] \right) \end{aligned} \tag{55}$$

subject to

$$Q_{R,s} = \quad \alpha \left( Q_{R,s-1} \cdot \beta + \Theta_{M,s} \right) - (V_s - V_{s-1}) \qquad \forall s = 2..R \tag{56}$$

$$Q_{R,1} = \quad \alpha \left( Q_{R,R} \cdot \beta + \Theta_{M,1} \right) - V_1 \tag{57}$$

$$\sum_{s=1}^{R} Q_{R,s} = \quad \lambda \alpha \beta \tag{58}$$

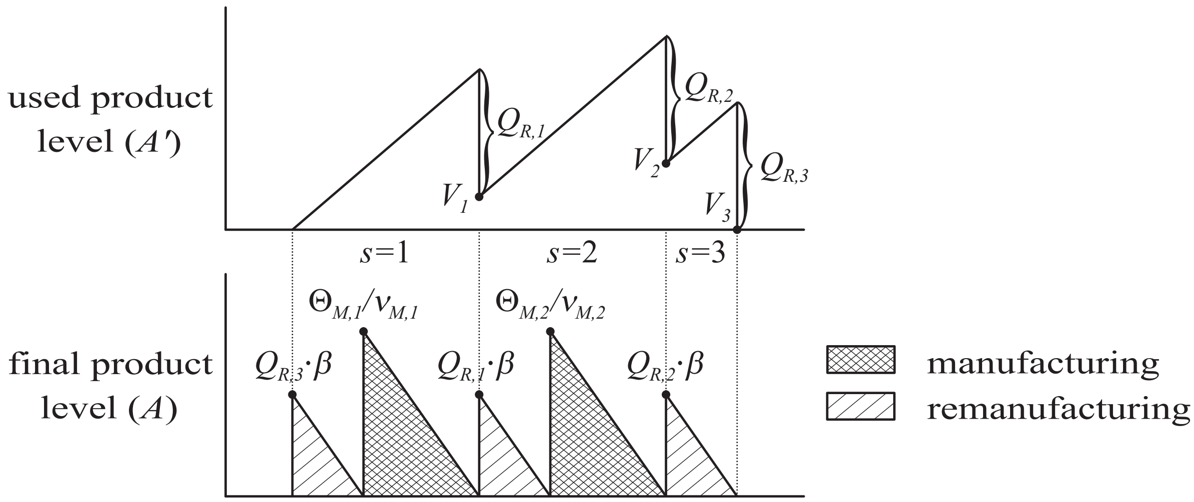


Figure 8: Exemplary cycle with $R = 3$ and $M = 2$

$$\sum_{s=1}^{R} \Theta_{M,s} = \lambda (1 - \alpha\beta) \tag{59}$$

$$\sum_{s=1}^{R} \nu_{M,s} = \max(R, M) \tag{60}$$

$$\sum_{s=1}^{R} \gamma_{M,s} = M \tag{61}$$

$$\Theta_{M,s} \leq \lambda\gamma_{M,s} \qquad \forall s = 1..R \tag{62}$$

$$\nu_{M,s} \geq 1 \quad \text{and integer} \qquad \forall s = 1..R \tag{63}$$

$$\gamma_{M,s} \geq 0 \quad \text{and integer} \qquad \forall s = 1..R \tag{64}$$

$$Q_{R,s}, \Theta_{M,s}, V_s \geq 0 \qquad \forall s = 1..R \tag{65}$$

The objective function $HC_1$ (55) represents the holding cost of both inventories for a cycle length of one time unit and needs to be minimized. Beginning with the final product level, the relevant area of this inventory has to be calculated which consists of $R + M$ right-angled triangles. Because of the imperfect remanufacturing process, only the fraction $\beta$ of all remanufactured products meets the requested quality standards to be resold to the customers. Therefore, each subcycle's remanufacturing batch satisfies customer demand for $Q_{R,s} \cdot \beta/\lambda$ time units. On the other hand, the amount to be manufactured in subcycle $s$ is divided into $\nu_{M,s}$ lots of equal size. Each lot contains, thus, $\Theta_{M,s}/\nu_{M,s}$ products to fulfill the demand for a period of $\Theta_{M,s}/(\nu_{M,s} \cdot \lambda)$ time units. Therefore, the area of the final product inventory is computed by

$$\sum_{s=1}^{R}\left[\frac{1}{2}\cdot(Q_{R,s}\cdot\beta)\cdot\frac{(Q_{R,s}\cdot\beta)}{\lambda}+\nu_{M,s}\cdot\frac{1}{2}\cdot\frac{\Theta_{M,s}}{\nu_{M,s}}\cdot\frac{\Theta_{M,s}}{\nu_{M,s}\cdot\lambda}\right]$$
$$=\frac{1}{2\lambda}\cdot\sum_{s=1}^{R}\left[(Q_{R,s}\cdot\beta)^2+\frac{(\Theta_{M,s})^2}{\nu_{M,s}}\right].$$

Analyzing the used product level is more complicated due to the possibility of having an initial inventory ($V_{s-1}$) as well as a final inventory of used products ($V_s$) in each subcycle $s$. Therefore, the area to be analyzed can take on a trapezoidal shape when both $V_{s-1}$ and $V_s$ are positive (as for the second subcycle of Figure 8). The relevant area is then computed by the general formula $0.5\cdot(a+b)\cdot h$ in which $h$ is the trapezoid's height, and $a$ as well as $b$ represent the lengths of its parallel sides. For subcycle $s$ the parallel sides are $V_{s-1}$ and $Q_{R,s}+V_s$, respectively. The trapezoid's height is equal to the subcycle's duration. As the number of products returning to the OEM in a subcycle is defined by $Q_{R,s}+V_s-V_{s-1}$, a subcycle lasts for $(Q_{R,s}+V_s-V_{s-1})/\lambda\alpha$ time units. The area of the used product inventory can, thus, be computed for each subcycle by:

$$\frac{1}{2}\left(V_{s-1}+Q_{R,s}+V_s\right)\cdot\frac{Q_{R,s}+V_s-V_{s-1}}{\lambda\alpha}$$
$$=\frac{1}{2\lambda\alpha}\left(V_{s-1}Q_{R,s}+V_{s-1}V_s-V_{s-1}^2+Q_{R,s}^2+Q_{R,s}V_s-Q_{R,s}V_{s-1}+V_sQ_{R,s}+V_s^2-V_sV_{s-1}\right)$$
$$=\frac{1}{2\lambda\alpha}\left((Q_{R,s}+V_s)^2-(V_{s-1})^2\right). \tag{66}$$

Due to the overall cyclic structure, equation (66) has to be adapted for the first subcycle. In this case, the predecessor of the first subcycle would be the last subcycle of the preceding cycle. Therefore, equation (66) becomes $\frac{1}{2\lambda\alpha}\cdot(Q_{R,1}+V_1)^2$ for $s=1$ since $V_R$ is set to zero. By multiplying each area with the respective holding cost parameter and summing up over all subcycles, the objective function (55) is established. It represents for a given policy structure (i.e. $R$ and $M$ are preset) the holding cost $HC_1$.

In order to guarantee feasibility of the solution, constraints (56) to (65) have to be met. The restrictions in (56) represent the inventory balance constraints of the used product level. They describe the inventory at the end of subcycle $s$ ($V_s$) as the inventory at its beginning ($V_{s-1}$) plus its inflows and minus its outflows. The inflows include all used products arriving in this subcycle. As subcycle $s$ has a length of $(Q_{R,s-1}\cdot\beta+\Theta_{M,s})/\lambda$ time units and $\lambda\alpha$ used products arrive per time unit, altogether $\alpha\cdot(Q_{R,s-1}\cdot\beta+\Theta_{M,s})$ used products reach the OEM in subcycle $s$. The outflows are computed by the remanufacturing lot initiated at the end of subcycle $s$ which comprises $Q_{R,s}$ products to be remanufactured. Using these inventory balance equations, one can derive constraint (56) after the following manipulation

$$V_s=V_{s-1}+\alpha\cdot(Q_{R,s-1}\cdot\beta+\Theta_{M,s})-Q_{R,s}$$
$$Q_{R,s}=\alpha\left(Q_{R,s-1}\cdot\beta+\Theta_{M,s}\right)-(V_s-V_{s-1}).$$

Constraint (57) has to be incorporated to reflect the cyclic structure of the underlying problem, i.e. a cycle's last subcycle is the predecessor of the successive cycle's first subcycle. Constraint (58) guarantees that all products returning during a cycle are remanufactured. Since demand cannot be met solely by remanufacturing, restriction (59) assures the missing components to be manufactured.

To apply the subcycle-oriented optimization approach, the number of remanufacturing and manufacturing lots has to be fixed in advance. As $R$ can be smaller than $M$, not all subcycles have to include a manufacturing lot. If, for instance, no manufacturing lot is scheduled in subcycle $s$, the value $\nu_{M,s}$ will be zero which would make the objective function infeasible (division by zero). In this case, constraints (60) to (64) ensure that no new product is fabricated, i.e. $\Theta_{M,s} = 0$. Moreover, $\nu_{M,s}$ is forced to be equal to one to avoid division by zero in the objective function. Forcing $\Theta_{M,s}$ and $\nu_{M,s}$ to zero and one, respectively, can be achieved by introducing another integer decision variable $\gamma_{M,s}$ that decides whether a subcycle contains a manufacturing lot or not. If not, $\gamma_{M,s}$ is zero and constraint (62) restricts $\Theta_{M,s}$ to be zero. Otherwise, if a subcycle contains at least one manufacturing lot, $\gamma_{M,s}$ can take on any positive integer value as it does not affect the objective function. Yet, restriction (61) ensures the sum of $\gamma_{M,s}$ over all subcycles to equate $M$. This combined with constraint (60) guarantees that at least $R - M$ (for $R > M$) subcycles do not contain a manufacturing batch. Constraints (63) to (65) restrict all decision variables to non-negative values. While this is sufficient for $Q_{R,s}$,$\Theta_{M,s}$, and $V_s$, the remaining variables $\nu_{M,s}$ and $\gamma_{M,s}$ have to be integer in addition. Furthermore, to ensure validity $\nu_{M,s}$ must be greater or equal to one.

The subcycle-oriented optimization approach can be applied to a multitude of policy structures to compute a benchmark solution. In order to do that, a number of non-linear optimization problems has to be solved. Although all constraints are linear, the objective function is non-linear because of the $\nu_{M,s}$ decision variables in its denominator. Therefore, a standard linear solver cannot be applied to generate the benchmark solution. Instead, the software package GAMS provides a number of solvers that can handle non-linearity quite efficiently. To determine the benchmark solution, the mixed-integer non-linear programming solver SBB has been applied. This solver uses a combination of the Branch&Bound methodology known from linear programming combined with one of the GAMS NLP solvers (for further details on the SBB solver please refer to SBB, 2009). With respect to time and solution quality, the NLP solver CONOPT3 worked best for this problem setting (see Drud, 2009, for additional details). As no NLP solver can guarantee to find the optimal solution to the NLP relaxations (the integrality constraint is relaxed), it cannot be proven that an optimization run provides the true optimal solution to a problem. Nevertheless, it is possible to compare the results of the predefined policy structures $(R, 1)$, $(1, M)$, and $(R, 1)^g$ with this benchmark solution to get an idea on their performance. To evaluate the potential benefits the benchmark solution is able to offer in comparison to the predefined policy structures, a numerical study is presented in the following Section 4.

# 4 Numerical study

Comparing the predefined policy structures with the benchmark solution in a numerical study requires appropriate test instances. Rardin and Uzsoy (2001) describe four different options on how to generate test instances properly. First, they name real world data sets as a viable source of information. Data sets taken from real applications promise the most realistic evaluation of the tested algorithms as all conclusions drawn from the experiments can be almost directly transferred to the real application. However, there are several pitfalls concerning real data sets. For example, gathering this kind of data can be extraordinarily difficult. In our problem context, estimating setup and holding cost parameters is sometimes a challenging task in a real-life environment. Furthermore, as there is only a limited number of real-life problems, the considered algorithms cannot be tested extensively with a large number of different parameter sets. Hence, Rardin and Uzsoy name random variants of real world data sets as a second source of generating problem instances. By maintaining most of the structural properties, the random variation of one or several parameters avoids the pitfall of not having enough real-life parameter sets. If no practical data is available at all, the third option of exploiting published and online libraries becomes interesting. Although being a rich source of different test instances for some problem settings, it may occur that a new algorithm is proposed performing only well on these instances. Thus, it must be ensured that a large number of different instances is tested with a new algorithm. Finally, a random instance generation provides the simplest and fastest way to generate a huge number of test instances. This fourth option becomes interesting when none of the other options (exclusive or combined) is able to establish a comprehensive set of experiments.

The numerical study conducted in this section uses the second methodology, variation of real world data. Yet, regarding this problem setting there are only a few contributions in literature presenting practical data. Tang and Teunter (2006), for instance, analyze the operations of a company that (re)manufactures water pumps for diesel engines. They provide data on five different types of water pumps including setup and holding cost parameters. In another contribution, Ashayeri et al. (1996) present the case of remanufacturing computers. As well, they illustrate a practical example including setup and holding cost parameters. Their example is taken as a base case scenario in this section. To evaluate the influence of all parameters, the base case scenario is modified in a sensitivity analysis afterwards. Of course, the data published in Tang and Teunter could have been taken as well for a base case scenario. At the end of this section, a short analysis of this data set is presented and compared to the results of the base case scenario. Interestingly, the important parameter constellation describing the ratio of $h_M$ to $h_R$ is the same in both contributions, i.e. holding a final product for one time unit in stock is twice as expensive as holding a returned product in the used product inventory. However, both contributions discuss that determining $h_R$ is especially difficult for practical applications.

***Base case scenario of Ashayeri et al.***

Ashayeri et al. present the following parameters which have been used for the base case scenario. The OEM faces a constant and continuous demand of 100 products per time unit which comprises in this case 3 days. Initiating a remanufacturing batch costs 50 Dutch guilders while setting up a manufacturing lot is a little more expensive with 150 guilders. Holding a used computer for three days costs 1 guilder, while holding a new or remanufactured computer costs 2 guilders. For the sake of simplicity, the currency is omitted in the following analysis. Over the infinite planning horizon, 60 % of the demand per time unit is returned to the computer remanufacturer. As Ashayeri et al. did not include a possible yield loss from remanufacturing, they assumed that all returns can be successfully remanufactured. In order to incorporate an imperfect remanufacturing process, we set $\beta$ to 80% for the base case scenario. This change to the original Ashayeri et al. scenario can be imposed as this parameter is altered later on to observe its influence on the performance of the preset policy structures compared to the benchmark solution. Table 2 summarizes all base case parameters. Applying

Table 2: Base case parameters

| $\lambda$ | $\alpha$ | $\beta$ | $K_R$ | $K_M$ | $h_R$ | $h_M$ |
|---|---|---|---|---|---|---|
| 100 | 60 % | 80 % | 50 | 150 | 1 | 2 |

equations (30) and (37) provides the cost minimizing parameters $R^*$ and $M^*$ for the predefined $(R, 1)$ and $(1, M)$ policy structures. We obtain

$$M^* = \sqrt{\frac{1}{4} + \frac{K_R \cdot (1-\alpha\beta)^2 \cdot h_M}{K_M \cdot (\alpha h_R + \alpha^2\beta^2 h_M)}} \updownarrow= 0.648 \updownarrow= 1 \quad \text{and}$$

$$R^* = \sqrt{\frac{1}{4} + \frac{K_M \alpha^2 \beta \cdot (h_R + h_M \beta)}{K_R \cdot \left(\alpha (1-\alpha\beta) h_R + (1-\alpha\beta)^2 h_M\right)}} \updownarrow= 1.698 \updownarrow= 2.$$

While a $(1,1)$ policy is the cost minimizing of all $(1,M)$ policies for the base case scenario, the best $(R,1)$ policy structure would be a $(2,1)$ policy. As elaborated in Section 2.5, the $(2,1)$ policy has to outperform the (1,1) policy. Thus, by using equations (27) and (28) the optimal cycle length $T^*_{R1}(2)$ and the optimal total cost $TC^*_{R1}(2)$ are computed

$$T^*_{R1}(2) = \sqrt{\frac{2 \cdot (R \cdot K_R + K_M)}{\lambda \cdot \left(\left(1 + \alpha\beta\left(\frac{1}{R} - 1\right)\right) \cdot \alpha h_R + \left(\frac{\alpha^2\beta^2}{R} + (1-\alpha\beta)^2\right) \cdot h_M\right)}}$$

$$T^*_{R1}(2) = 2.0185 \text{ time units} \approx 6 \text{ days}$$

$$TC^*_{R1} = \sqrt{2\lambda \cdot (RK_R + K_M) \cdot \left(\left(1 + \alpha\beta\left(\frac{1}{R} - 1\right)\right)\alpha h_R + \left(\frac{\alpha^2\beta^2}{R} + (1-\alpha\beta)^2\right) h_M\right)}$$

$$TC^*_{R1}(2) = 247.71$$

Regarding the predefined $(R,1)^g$ policy with geometrically decreasing remanufacturing batches, the cost minimizing number of remanufacturing lots $R^*$ is determined by equation (46). As no closed-form expression exists to compute $R^*$, a local search procedure has been proposed to determine this value. This procedure begins to compute the total cost for $R = 1$. Thereafter, the total cost value is computed for $R + 1$ until the total cost increases for the first time. When this happens, the local search procedure terminates. The total cost function for an $(R,1)^g$ policy is

$$TC^*_{R1^g} = \sqrt{2\lambda\left(RK_R + K_M\right)\left(\left(\alpha h_R + \alpha^2\beta^2 h_M\right)\left(\frac{1-\alpha\beta}{1+\alpha\beta} \cdot \frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}\right) + h_M\left(1-\alpha\beta\right)^2\right)}$$

Table 3 presents the results for all $R$ values between one and five. It can be seen that the total cost value for $R$=2 is the smallest with 238.40 and that it constantly increases for $R \geq 2$. Therefore, the best $(R,1)^g$ policy is a $(2,1)^g$ policy.

Table 3: Total cost values for the base case scenario for $1 \leq R \leq 5$

| $R$=1 | $R$=2 | $R$=3 | $R$=4 | $R$=5 |
|---|---|---|---|---|
| 253.11 | 238.40 | 245.71 | 258.60 | 273.20 |

By switching to geometrically decreasing remanufacturing lots, a cost saving of around 3.91% ($\frac{247.71-238.40}{238.40}$-1) is realized. The relevant decision variables are summarized in Table 4. The optimal cycle length for the $(2,1)^g$ policy is a little longer than for the (2,1) policy with remanufacturing lots of equal size. When applying a $(2,1)$ policy all returns in a cycle are remanufactured in two equal lots, i.e. $Q_{R,1} = Q_{R,2} = 0.5 \cdot \lambda\alpha T$. On the other hand, the remanufacturing lot sizes of the $(2,1)^g$ policy are geometrically decreasing and can be computed according to equations (41). For both policy structures, the only manufacturing lot in a cycle comprises exactly $\lambda(1-\alpha\beta)T$ newly fabricated products.

Table 4: Decision variables for the $(2,1)$ and $(2,1)^g$ policy structures

| | (2,1) policy | $(2,1)^g$ policy |
|---|---|---|
| $T$ | 2.0184 | 2.0973 |
| $Q_{R,1}$ | 60.552 | 85.0257 |
| $Q_{R,2}$ | 60.552 | 40.8123 |
| $Q_{M,1}$ | 104.9568 | 109.061 |

We omit to present the results of the $(2,1)$ and $(2,1)^g$ policies graphically as they correspond to the inventory developments of Figure 7. In order to evaluate whether these policies obtained good solutions, the benchmark solution to the base case has been calculated as well. Altogether, 100 different combinations of $R$ and $M$ have been analyzed in which each parameter could take on integer values between 1 and 10. The result has been that the benchmark solution obtained a solution corresponding to the $(2,1)^g$ policy's solution and is, thus, not able to improve the solution obtained by the predefined policies.

The remainder of this subsection presents the results of a one parameter sensitivity analysis. While keeping six of the seven parameters (please refer again to Table 2 for a short overview) constant, the residual parameter is altered in a reasonable range. This sensitivity analysis is conducted for all parameters except $\lambda$ since this parameter does not influence the solution structure which can be seen in the benchmark solution's objective function (55). At first, the influence of the return fraction $\alpha$ is examined.

***Return fraction*** $\alpha$

The fraction of used products returning to the OEM $\alpha$ can vary theoretically between 0 % and 100%. As the extreme values do not seem to be reasonable since the entire demand would be satisfied by either remanufacturing or manufacturing only, the sensitivity analysis considers all $\alpha$ values between 1 % and 99 % in steps of 0.5%. The three predefined policy structures $(R,1)$, $(1,M)$, and $(R,1)^g$ have been tested with this data and the minimum total cost value for each preset policy structure is presented graphically in Figure 9. There, the best preset policy structure is indicated below the minimum total cost of all policies.

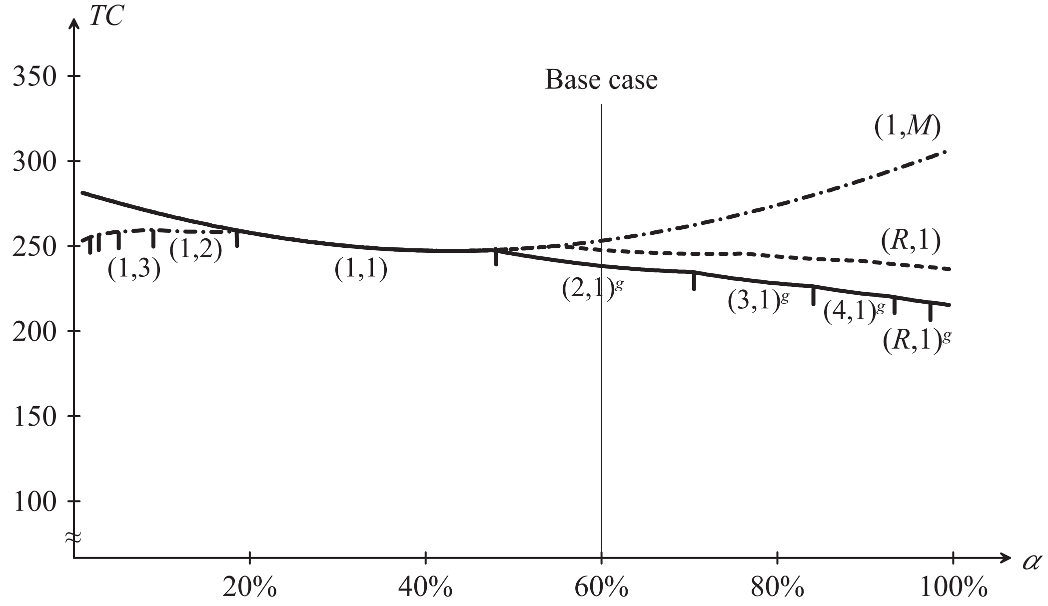


Figure 9: Minimum total cost of the preset policy structures for different $\alpha$ values

For small values of $\alpha$ ($\alpha \leq 19\%$), the $(1, M)$ policy dominates the structures that propose to initiate more than one remanufacturing lot in a cycle. If $\alpha$ is low, there are not many returns to remanufacture. Hence, a large fraction of customer demand needs to be satisfied by manufacturing product $A$. Since all returned products have to be remanufactured as the option to dispose them of is prohibited, the cycle length is quite long to collect a sufficient number of returns to remanufacture in a single batch. Thus, the total amount to be manufactured increases which lets the number of manufacturing lots become larger in a cycle as well. By exploiting condition (39) the exact value of $\alpha$ is calculated at which the optimal number of manufacturing lots switches from two to one. When $\alpha$ is smaller than 19.18% it is better to schedule two manufacturing lots instead of one in a cycle[17].

For $\alpha$ between 19.18 % and around 48 % all preset policies determine the same minimum total cost. In this range, a $(1, 1)$ policy is the best choice for all preset policy structures, i.e. they coincide. For $\alpha$ larger than 48 % the $(R, 1)^g$ policy dominates the other policy structures. Therefore, condition (47) provides the exact $\alpha$ value for which an $(R, 1)^g$ policy begins to outperform both the $(1, M)$ and $(R, 1)$ policies. By applying the bisection method to this method an exact $\alpha$ of 47.65 % has been computed.

In order to evaluate whether the total cost values determined by the preset policy structures might be far from optimal, the benchmark solution has been obtained for all problem instances as well. Since the optimization approach requires that $R$ and

[17]For details on how to determine this value, please refer to the Appendix, page 61

$M$ are set in advance, the number of examined combinations has to be limited to keep the computational effort controllable. Altogether, the mixed-integer non-linear optimization problem has calculated the benchmark solution for 36 policy structures where $R$ as well as $M$ could take on any integer value between 1 and 6. By restricting the number of combinations and as no NLP solver can guarantee to provide the true optimal solution to a problem, it cannot be guaranteed to find the optimal solution to the entire problem. However, this approach offers an opportunity to evaluate the performance of the preset policies on a general level what is not found in literature up to now.

The benchmark solution is able to improve the preset policies' solutions in some cases but not in general. In order to elaborate the influence of equal remanufacturing lots, the benchmark solution is compared on the one hand to the minimum total cost of the $(R, 1)$ and $(1, M)$ policies. On the other hand, the benchmark solution is confronted with the best result from the $(R, 1)$, the $(1, M)$, and the new $(R, 1)^g$ policy. Figure 10 presents these results.

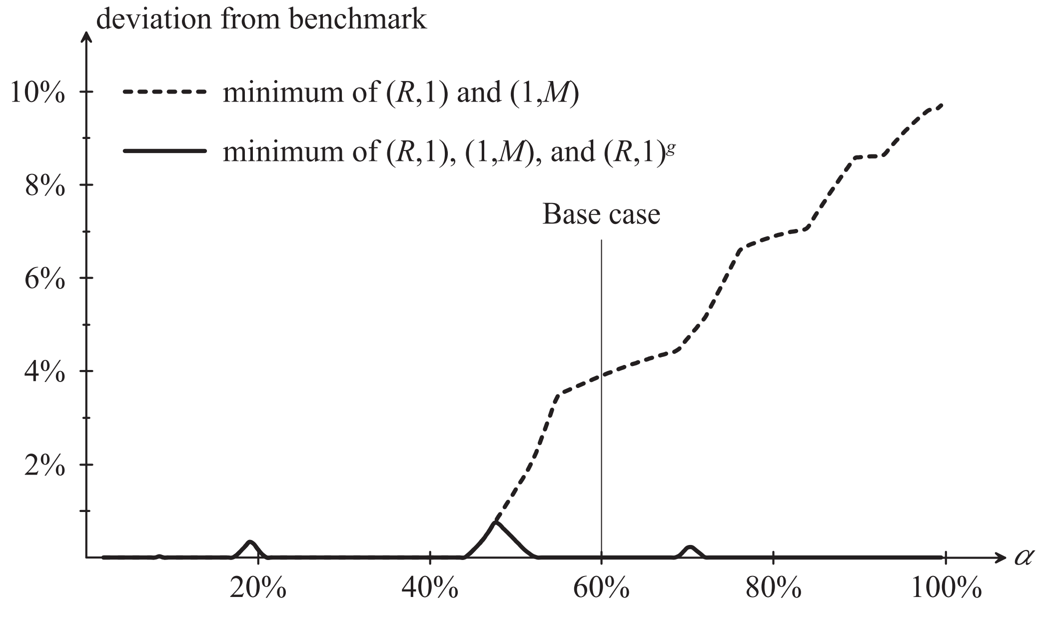


Figure 10: Deviation from benchmark solution for different $\alpha$ values

Applying the $(R, 1)$ and $(1, M)$ policy structures leads to an error of more than 9% for large return fractions when compared to the benchmark. Furthermore, by including the $(R, 1)^g$ policy structure into the decision making process, the deviation from the benchmark solution can be limited to less than 1% over all instances tested for a variation of $\alpha$ for the base case. For instance, the maximum deviation of the best preset policy structure to the benchmark solution has been around 0.75% for a return fraction of 47.5%. For this return fraction, the benchmark solution proposed a (3,2)

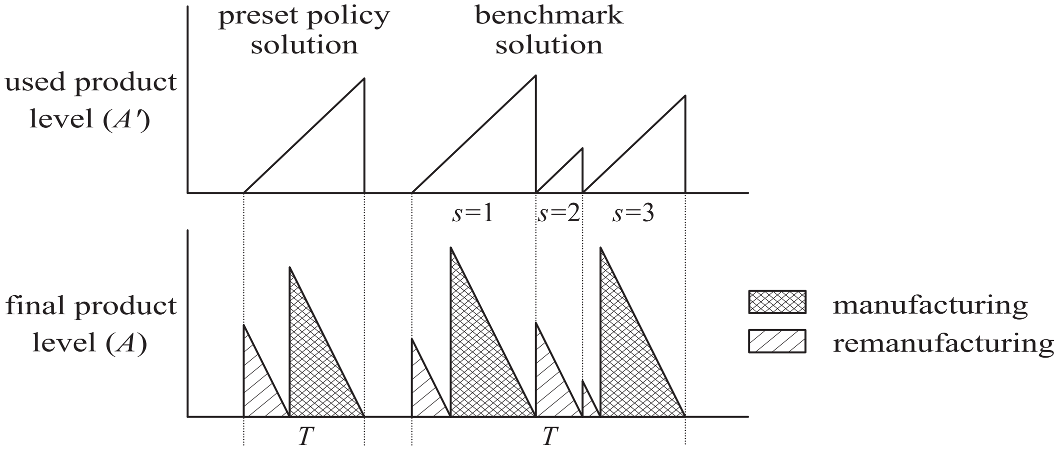


Figure 11: $(1,1)$ and $(3,2)$ policies for base case with $\alpha$=0.475

policy while a $(1,1)$ policy has been the best suggestion by the preset policies. The proposed cycles of the $(1,1)$ and $(3,2)$ policies are depicted in Figure 11 while their relevant decision variables are presented in Table 5.

Table 5: Decision variables for the $(1,1)$ and $(3,2)$ policies

| | preset policy structure $(1,1)$ policy | benchmark solution $(3,2)$ policy |
|---|---|---|
| $TC$ | 247.59 | 245.76 |
| $T$ | 1.6155 | 3.6621 |
| $Q_{R,1}$ | 76.7363 | 78.7352 |
| $Q_{R,2}$ | / | 29.9194 |
| $Q_{R,3}$ | / | 65.2952 |
| $Q_{M,1}$ | 100.161 | 113.5251 |
| $Q_{M,2}$ | / | / |
| $Q_{M,3}$ | / | 113.5251 |

The most striking difference between both solutions is their divergent cycle length. It can be seen that the $(1,1)$ policy's cycle length is much shorter than the cycle length for the benchmark $(3,2)$ policy. This is because the benchmark solution needs to divide the much larger setup cost of scheduling and therefore needs a longer cycle to do this efficiently. Comparing the total cost values between both solutions, the $(1,1)$ policy obtained a total cost of 247.59 per time unit while the $(3,2)$ structure is able to reduce the total cost to 245.76 per time unit. The relative deviation between both values is, thus, around 0.75%. When analyzing the benchmark in greater detail, the $(3,2)$

solution can be separated into two smaller cycles. The first smaller cycle consists of $Q_{R,3}$, $Q_{R,1}$, and $Q_{M,1}$ which coincide with a $(2,1)^g$ policy structure, i.e. $Q_{R,2} = \alpha\beta \cdot Q_{R,1}$. Thereafter, $Q_{R,2}$ and $Q_{M,3}$ correspond to a $(1,1)$ policy. Moreover, the manufacturing lots of both smaller cycles are of equal size.

Interestingly, the deviation of the best preset policy to the benchmark solution follows a characteristic pattern. For a multitude of instances, at least one of the preset structures obtains the benchmark solution. This might not be the case when the overall optimal solution to each instance could be obtained which is not possible due to its computational complexity. However, in some areas the benchmark solution is already better than the best preset policy structures of which four areas can be identified in Figure 10. Although hardly recognizable, the first return fraction for which this happens is 8.5%. Moreover, around the return fractions 18%, 45%, and 70% the other deviations can be found.

Without loss of generality, either the ratio of $R$ to $M$ or its inverse is an integer number for all preset policy structures as either $R$ or $M$ is one. Yet, if the benchmark solution deviates from these policy structures, both the ratio of the benchmark's $R$ and $M$ as well as its inverse are not integer. This fact has been depicted in Figure 12 which exhibits the benchmark solution's ratio of $R$ to $M$ depending on $\alpha$.

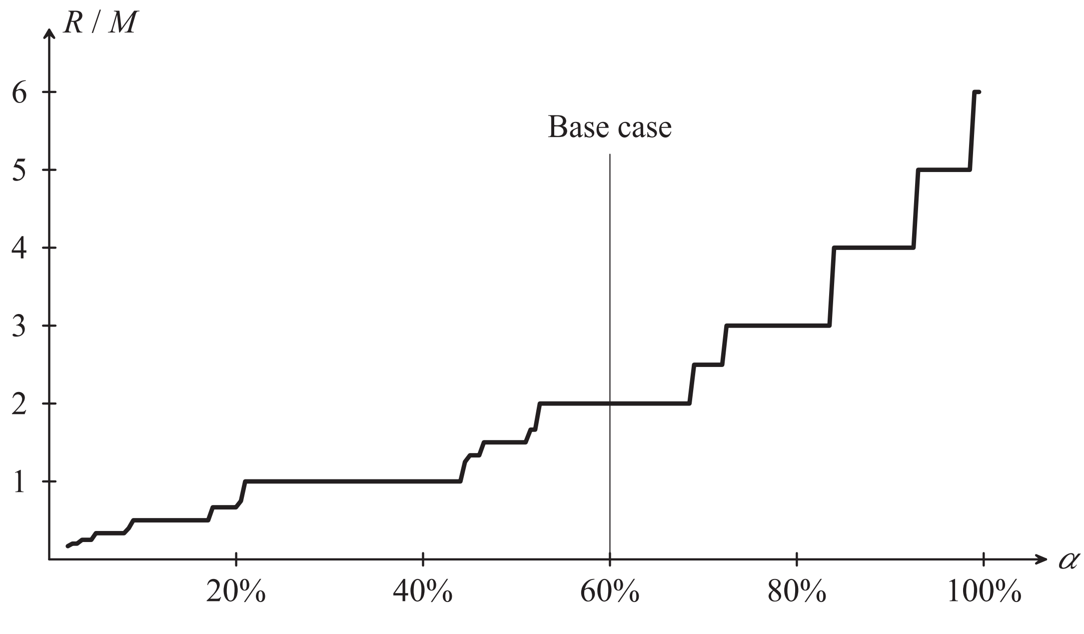


Figure 12: Ratio of $R$ to $M$ for the benchmark solution

It can be seen that the ratio of $R$ to $M$ never decreases when $\alpha$ becomes larger. For instance, around $\alpha = 45\%$ the value of this ratio is between one and two. Due to the experimental design (restricting the maximum value of $R$ and $M$ to six) only five different policy structures can be found that show a ratio between one and two:

the (3,2) [and therefore also the (6,4) policy which yields the same result], (4,3), (5,3), (5,4), and (6,5) policies. Except for the last one, all policy structures have been chosen by the benchmark solution for at least one $\alpha$ value. When varying the other parameters in the remaining sensitivity analysis, the same general monotonic behavior can be observed. This leads to the conjecture that the ratio of $R$ to $M$ for the exact solution increases monotonically when $\alpha$ increases. However, this conjecture cannot be proven since it is unlikely to determine the optimal solution, e.g., for a (201, 100) policy with the currently existing optimization software. Next, the influence of the yield parameter $\beta$ is analyzed.

***Yield parameter*** $\beta$

While keeping the remaining base case parameters constant, the fraction of successfully remanufactured products $\beta$ is altered in the following. This parameter has not been given by Ashayeri et al. and is, thus, of special interest. Like the return fraction, $\beta$ can be changed between 0 and 100 %. Yet, for the experiments $\beta$ is alternated between 1 and 100 % in steps of 0.5 %. Figure 13 presents the solutions of the preset policy structures.

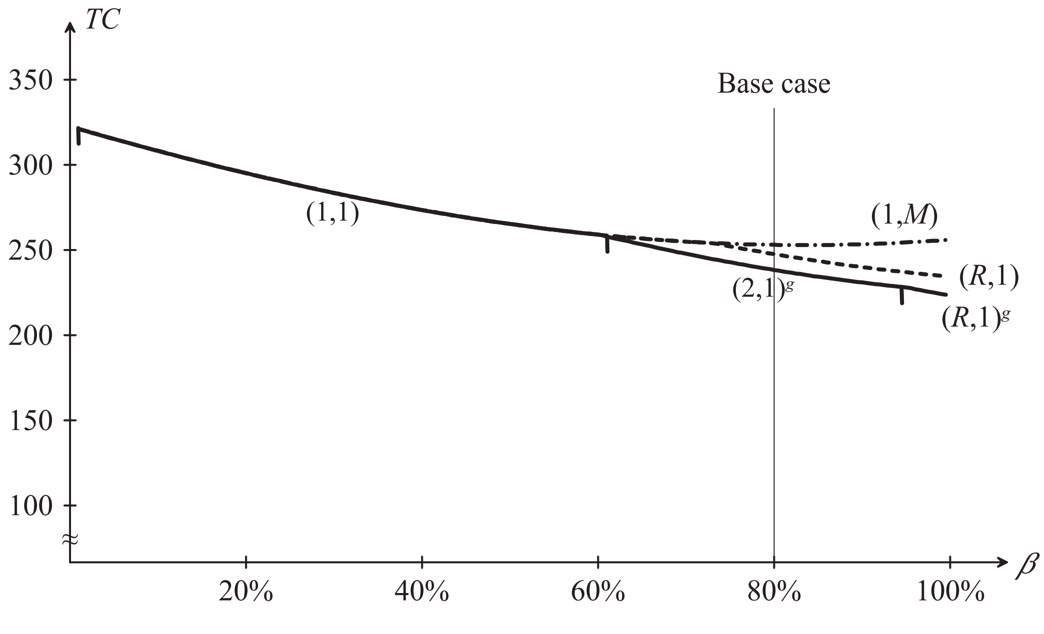


Figure 13: Minimum total cost of the preset policies for different $\beta$ values

For $\beta$ smaller than around 60% all preset policies determine the same result, i.e. a (1,1) policy is suggested. As $\beta$ become larger, the preset policy structures differ in their evaluation. Again, the $(R, 1)^g$ policy outperforms both the $(R, 1)$ and $(1, M)$ policies. Interestingly, the performance gain is largest for $\beta = 100\%$ which has been the original assumption of Ashayeri et al. Therefore, the declaration of an imperfect remanufacturing process as base case scenario has been reasonable. By analyzing condition (47) the exact value of $\beta$ is determined at which the $(R,1)^g$ policy's solution begins to yield a better result than the other two structures. Using the bisection method again derives a $\beta$ of 60.17%.

To compare the solutions of the preset policy structures to the benchmark solution, Figure 14 depicts the percentage error between both methodologies. The influence of a

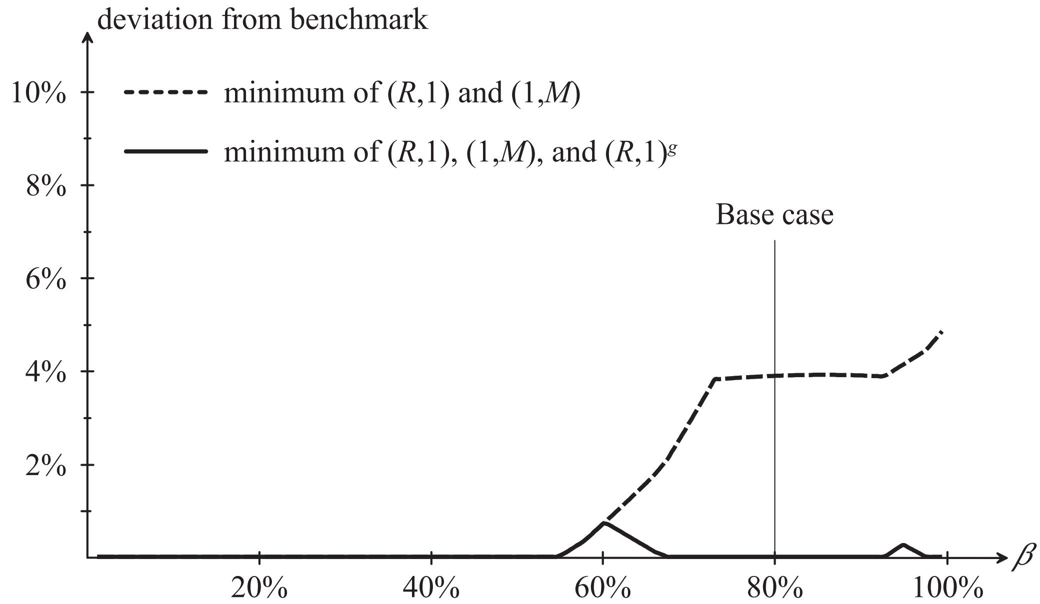


Figure 14: Deviation from benchmark solution for different $\beta$ values

large $\beta$ appears not to be as strong as for a large return fraction $\alpha$. Yet, a percentage error of more than 5 % compared to the benchmark solution can be observed when the decision maker omits to check the $(R,1)^g$ policy's solution. However, including the $(R,1)^g$ policy's solution does not always provide the best solution. In this analysis, there are two regions in which the benchmark solution is better than the preset structures. The first area can be found around $\beta$=60% at which the best preset policy structure changes from a (1,1) policy to a $(2,1)^g$ policy. Here, as well as for the return fraction $\alpha$, the benchmark obtains a solution for which the ratio of $R$ to $M$ lies between one and two. Consequently, the area around $\beta = 95\%$ shows the transition from a $(2,1)^g$ to a $(3,1)^g$ policy. In the following, the effects of diverse holding cost values are examined.

***Holding cost parameters*** $h_R$ ***and*** $h_M$

Regarding the holding cost parameters $h_R$ and $h_M$, not only the absolute values are of importance but also the ratio of both values. It has been observed in Table 1, for instance, that the ratio of the holding cost parameters determines whether an $(R,1)^g$ policy with geometrically decreasing remanufacturing lots finds a better solution than an $(R,1)$ policy with equal remanufacturing lots. At first, the influence of the holding cost for the used product inventory is examined. As $h_R$ must not exceed $\beta h_M$, it has been chosen to take on values between 0 and 1.6 in steps of 0.01. Figure 15 presents the best solutions obtained by the preset policy structures when $h_R$ is altered. In contrast

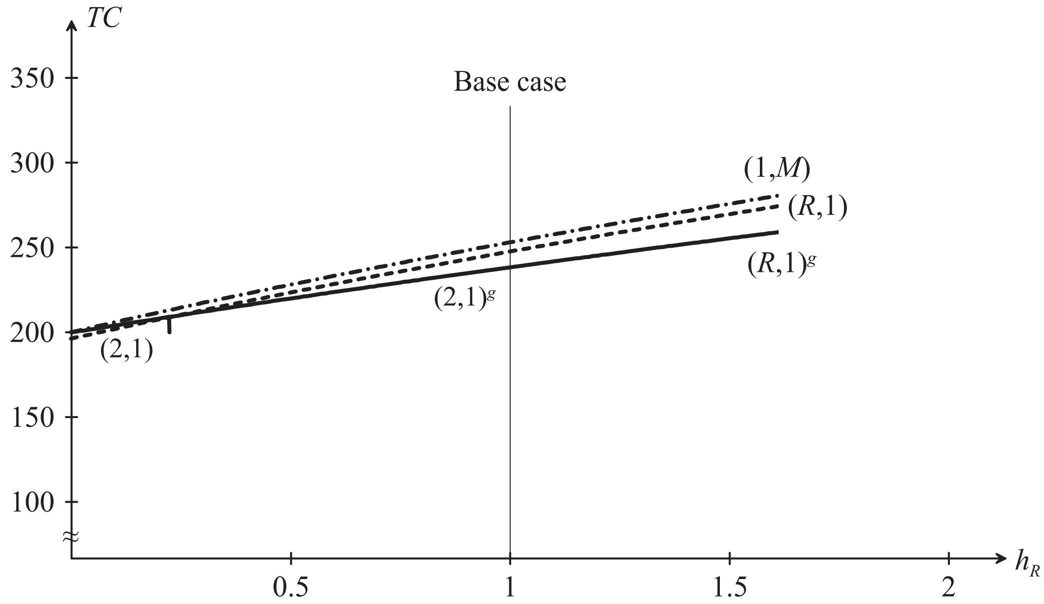


Figure 15: Minimum total cost of the preset policies for different $h_R$ values

to both Figure 9 for the return fraction $\alpha$ and Figure 13 for the quality parameter $\beta$ all preset policies determine a different solution, i.e. a (1,1) policy has never been the best proposed solution. Instead, the $(R,1)^g$ policy dominates both competitors for most of the $h_R$ values except for some small $h_R$ values. There, the $(R,1)$ policy is the best alternative. Considering the exact value of the change, it must be noticed that the switch takes place from a (2,1) to a $(2,1)^g$ policy. Therefore, the value of $h_R$ can be computed by exploiting condition (50) which gives an $h_R$ of 0.2391. This means if the holding cost rate for the used product level drops below 0.2391 (which is around 12 % of the final product level's holding cost), a policy structure with equal remanufacturing batches is preferable. As initiating equally sized remanufacturing lots reduces the inefficiency in the final product's stock, it is reasonable to take inefficiencies in the used product level into account when the holding cost $h_R$ is comparatively low.

The percentage error when confronted with the benchmark solution is presented in Figure 16. For $h_R$ smaller than 0.2391 both curves are identical as including the $(R,1)^g$ policy into the decision making process does not yield any benefit. However, if

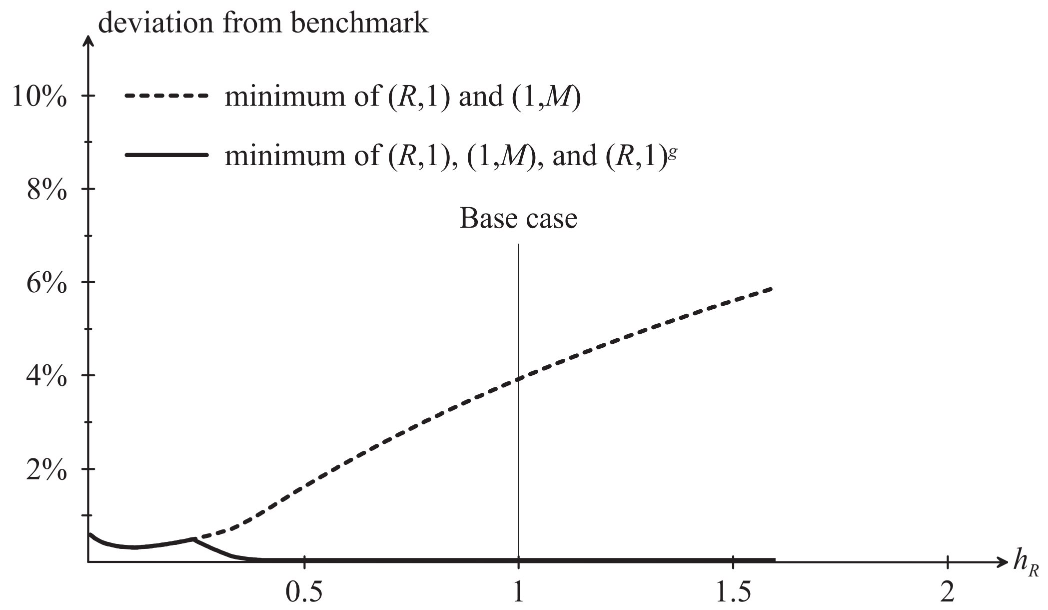


Figure 16: Deviation from benchmark solution for different $h_R$ values

$h_R$ approaches $\beta h_M$ the benefit becomes larger until it reaches around 6 % when they are almost identical. Furthermore, the $(R, 1)^g$ policy coincides with the benchmark solution for all $h_R$ values larger than 0.4 as the benchmark always computes a policy structure similar to a $(2, 1)^g$ policy.

Regarding the holding cost value for the final product inventory, similar conclusions can be drawn. Since $h_M$ must not be smaller than $h_R/\beta$ the smallest value for $h_M$ in this sensitivity analysis is 1.2. The maximum value, on the other hand, is set to be three. Within this range all values in steps of 0.01 have been examined. Figure 17 presents the results of the three preset policies that reflect the findings of Figure 15. Over all instances, the $(R, 1)^g$ policy provides the best results of the preset policy structures. To be more precise, the proposed policy has been a $(2, 1)^g$ policy for all tested $h_M$ values. Moreover, the absolute deviation between the $(R, 1)^g$ and the $(R, 1)$ policies is largest for $h_M$ values that lie close to $h_R/\beta$. As the total cost increase the larger $h_M$ becomes, the largest relative deviation is observed for small $h_M$ values, too. If the experiments would have been extended to incorporate $h_M$ values larger than 8.3647, the $(R, 1)^g$ policy would have been outperformed by the $(R, 1)$ policy. This value can be derived from the ratio of $h_M$ to $h_R$ in equation (50) that describes for a given value of $h_R$ the value of $h_M$ at which a $(2, 1)$ policy is better than a $(2, 1)^g$ policy. We omit to present the percentage error with respect to the benchmark for a varying $h_M$ as the results can be derived from Figure 17 as well. After analyzing the influence of both holding cost parameters, the influence of the setup cost parameters $K_R$ and $K_M$ is evaluated.

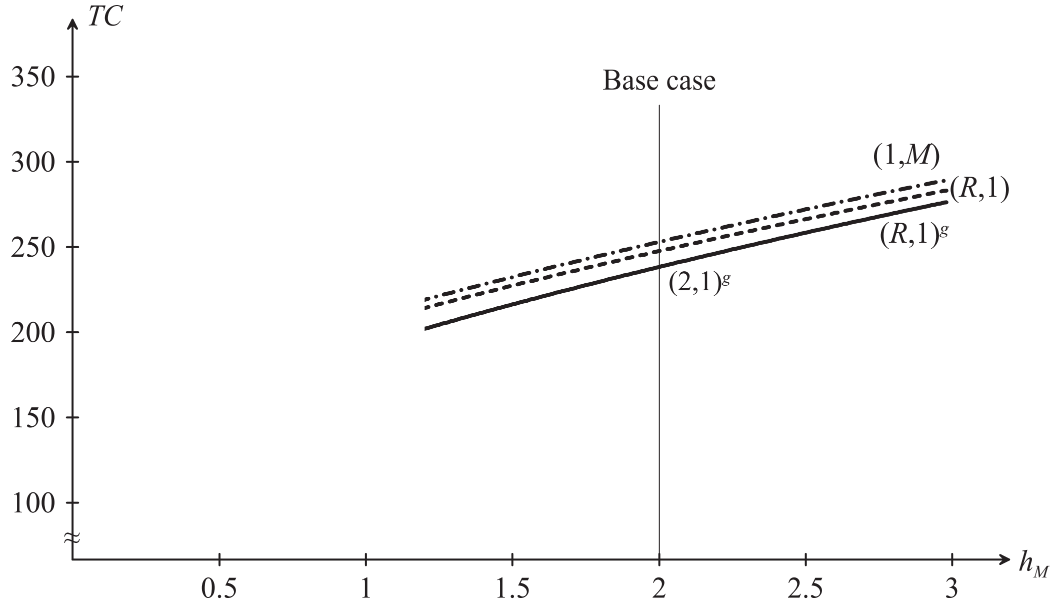


Figure 17: Minimum total cost of the preset policies for different $h_M$ values

***Setup cost parameters*** $K_R$ ***and*** $K_M$

Next to the holding cost parameters, the setup cost values have a direct influence on the number of lots in a cycle. In general, when the setup cost falls while keeping all other parameters constant, the number of lots does not decrease. To verify this general thought for the underlying problem, both setup cost parameters have been altered to take on values between 0 and 250 in steps of 1. Starting with the setup cost for remanufacturing $K_R$, the solutions of the preset policy structures have been depicted in Figure 18. For the variation of the base case scenario with respect to the setup cost for remanufacturing, two different phases can be observed. For $K_R$ being larger than 104, all preset policy structures compute the same result, a $(1,1)$ policy. If, on the other hand, $K_R$ is smaller than 104 the $(R,1)^g$ policy yields the minimum total cost of these policies. The exact value can be determined by exploiting condition (47) since this condition describes the transition from a (1,1) to a $(2,1)^g$ policy structure. In this case, the exact value for $K_R$ is 103.8156. Otherwise, by manipulating condition (39) the exact value of $K_R$ can be determined from which a $(1,2)$ policy dominates the $(1,1)$ policy. The value obtained thereby needs to be larger than 250 as the $(1,M)$ policy does not dominate the other two policy structures in Figure 18. The exact value is for the base case scenario a $K_R$ of 588.4615. A (1,1) policy is, therefore, the best preset policy structure for all $K_R$ values between 103.8156 and 588.4615.

In order to evaluate the overall solution quality of the preset policy structures, they

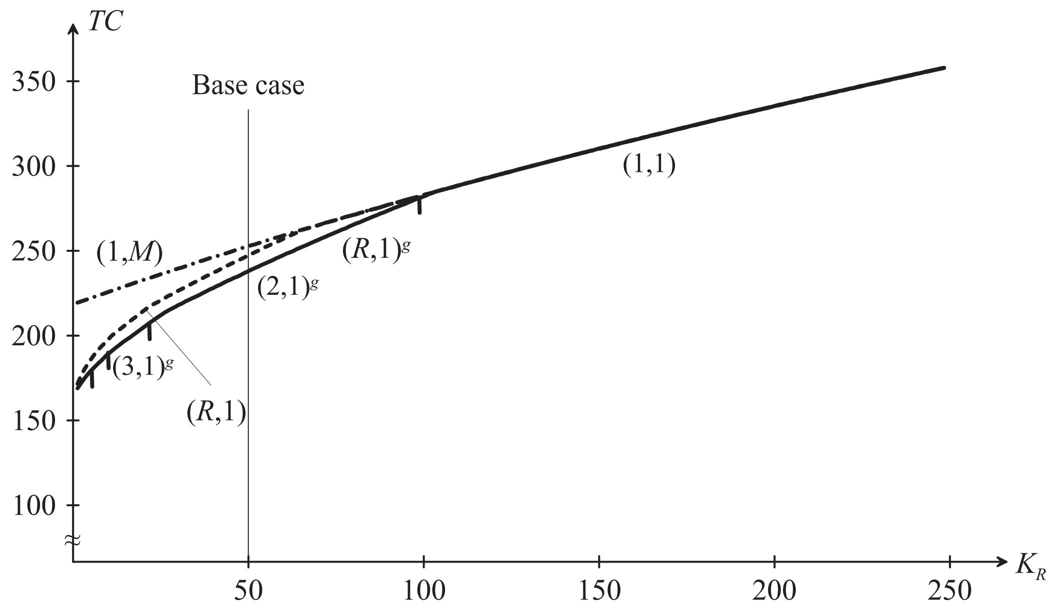


Figure 18: Minimum total cost of the preset policies for different $K_R$ values

are confronted with the benchmark solution as well. Figure 18 depicts the results.

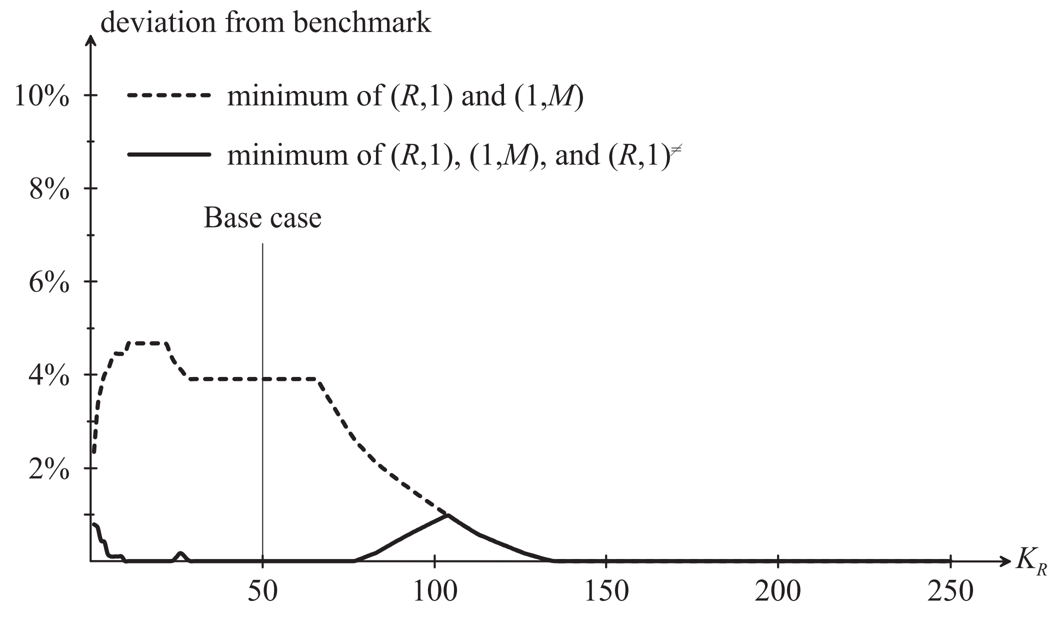


Figure 19: Deviation from benchmark solution for different $K_R$ values

As has been observed for all parameters until now, neglecting the opportunity to consider geometrically decreasing remanufacturing lots can lead to a significant error in the problem's solution. When varying $K_R$, this can be observed for a relatively small setup cost of remanufacturing. Yet, even when incorporating the $(R,1)^g$ policy, an error of up to 1% prevails when comparing the preset policy structures to the benchmark solution. Especially for $K_R$ between 77 and 134 this error is recognizable.

After elaborating the results for $K_R$, the analysis is put forth for the setup cost of initiating a manufacturing lot $K_M$. This parameter is altered as well between 0 and 250 in steps of 1. Figure 20 illustrates the results of the experiments regarding the minimum total cost for each preset policy structure. In this Figure, three different

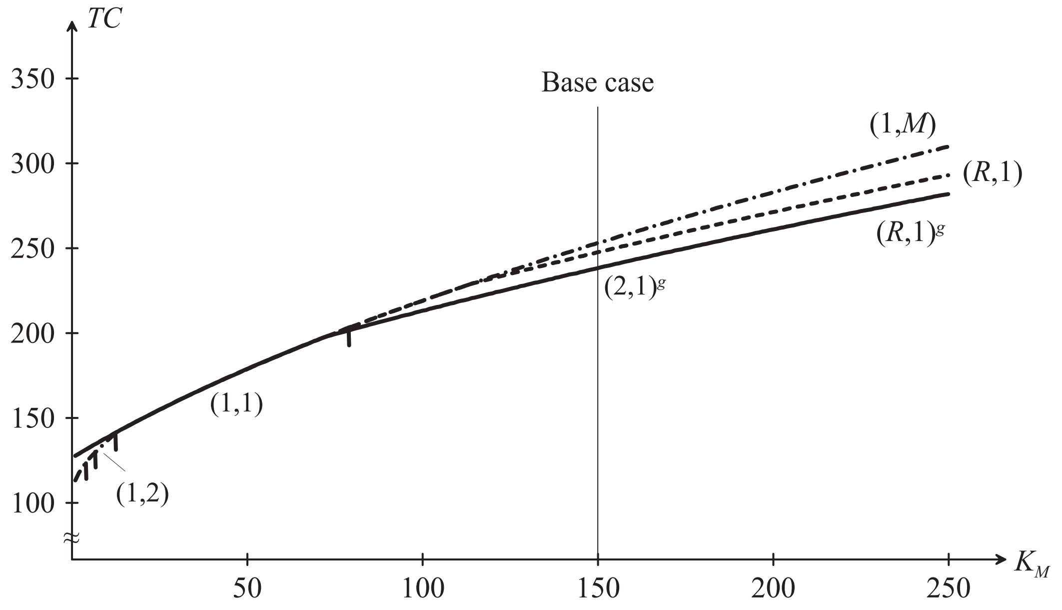


Figure 20: Minimum total cost of the preset policies for different $K_M$ values

sections can be found. When the setup cost for manufacturing is quite small, the $(1, M)$ policy calculates the best results since more than one manufacturing lot in a cycle is beneficial. The second section is represented by a (1,1) policy in which all preset policy structures determine the same result. Finally, the last section is characterized by the $(R, 1)^g$ policy dominating both the $(R, 1)$ and $(1, M)$ policy structures. The transition values that limit these sections can be determined by exploiting conditions (39) and (47). Solving these conditions with respect to $K_M$, condition (39) computes a $K_M$ of 12.7451 representing the transition from a $(1, 2)$ to a $(1, 1)$ policy. A $K_M$ of 72.2341, on the other hand, defines the transition from a $(1, 1)$ to a $(2, 1)^g$ policy. The second section of Figure 20 lies therefore between $K_M$=12.7451 and $K_M$=72.2341.

Concluding, the benchmark solution has been obtained as well for all instances regarding a variation of $K_M$ and can now be opposed to the preset policy structures in Figure 21. This figure presents an almost familiar picture. Omitting geometrically decreasing remanufacturing lots results in an error of up to 3.9 %. Interestingly, this percentage deviation has been constant over a multitude of instances from $K_M = 114$ to the upper bound. This is because the best preset policy structures have been the $(2, 1)^g$ and the $(2, 1)$ policies. Since the holding cost per time unit is not affected by

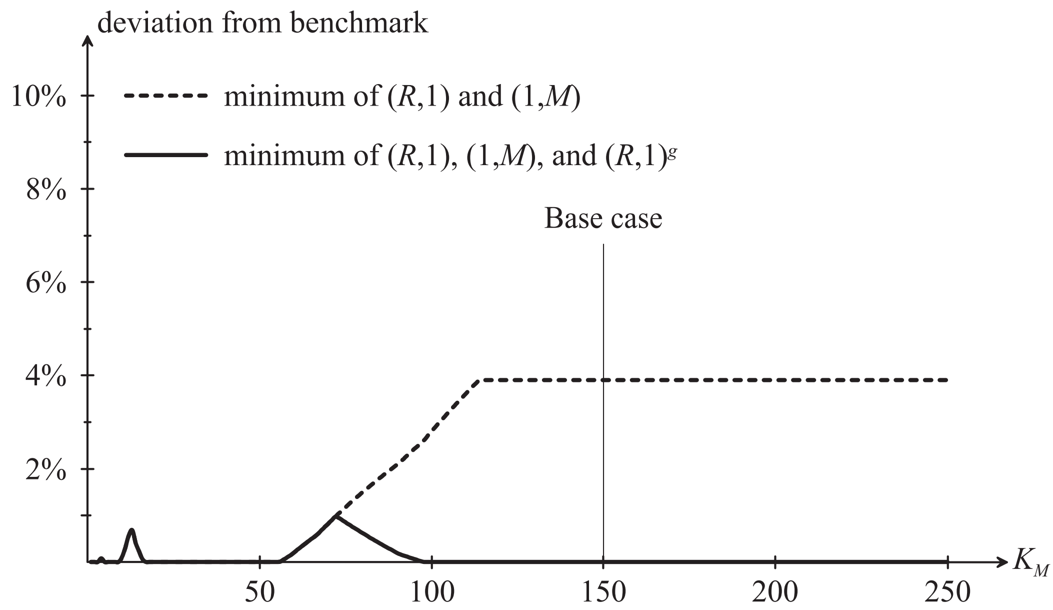


Figure 21: Deviation from benchmark solution for different $K_M$ values

a variation in the setup cost for manufacturing, the total cost value increases proportionally with an increasing $K_M$ value. As a matter of fact, this happens independently from the presumption of considering equal or different remanufacturing lots in a cycle. The same behavior could also be observed in Figure 19 for the setup cost for remanufacturing. Although not being as prominent as for the setup cost for manufacturing, the same explanation can be used there.

#### ***Parameter settings of Tang and Teunter***

Before concluding this chapter, the effects of a different base case scenario are analyzed in the following. As mentioned above, Tang and Teunter (2006) present real-life data for the (re)manufacturing process of water pumps for diesel engines. In their contribution, five different types of water pumps (denoted by TT1 to TT5) are considered. Table 6 summarizes the relevant setup and holding cost parameters for these products. For all products, the setup cost to initiate a (re)manufacturing batch is 20. Furthermore, holding a final product for one time unit costs twice the amount of holding a returned product for one time unit. The remanufacturer faces only a small return ratio of water pumps amounting to 20% for all analyzed products. Since the yield parameter $\beta$ has not been included by Tang and Teunter, we fix it to 80% as for the Ashayeri et al. base case. Finally, demand for TT1 to TT5 differs between 3 and 30 units per time unit.

In order to determine the best preset policy structure, equations (30), (37), and

Table 6: Parameters for TT1 to TT5

| Product | $\lambda$ | $\alpha$ | $\beta$ | $K_R$ | $K_M$ | $h_R$ | $h_M$ |
|---|---|---|---|---|---|---|---|
| TT1 | 9 | 20 % | 80 % | 20 | 20 | 0.0088 | 0.0175 |
| TT2 | 9 | 20 % | 80 % | 20 | 20 | 0.0132 | 0.0263 |
| TT3 | 9 | 20 % | 80 % | 20 | 20 | 0.0175 | 0.035 |
| TT4 | 30 | 20 % | 80 % | 20 | 20 | 0.0219 | 0.0438 |
| TT5 | 3 | 20 % | 80 % | 20 | 20 | 0.0263 | 0.0525 |

Table 7: Best preset policy structure and benchmark for TT1 to TT5

| | Best preset policy structure | | | Benchmark solution | | |
|---|---|---|---|---|---|---|
| Product | $R$ | $M$ | $TC$ | $R$ | $M$ | $TC$ |
| TT1 | 1 | 2 | 3.0087 | 2 | 5 | 3.0085 |
| TT2 | 1 | 2 | 3.6877 | 2 | 5 | 3.6873 |
| TT3 | 1 | 2 | 4.2524 | 2 | 5 | 4.2517 |
| TT4 | 1 | 2 | 8.6853 | 2 | 5 | 8.6839 |
| TT5 | 1 | 2 | 3.0075 | 2 | 5 | 3.0071 |

(46) are evaluated for all products. As a result, a $(1, 2)$ policy outperforms both the $(R, 1)$ and the $(R, 1)^g$ policy structures for all parameter sets examined. Afterwards, by applying the optimization approach presented in Section 3 the benchmark solution for all products is obtained. Due to the similar parameter structure, the benchmark solution coincides for all products as well, i.e. a policy with two remanufacturing and five manufacturing lots in a cycle is recommended. Yet, the error of applying a (1,2) policy instead of the benchmark solution is less than 0.01 % for all products. Table 7 summarizes the relevant results.

A sensitivity analysis has been conducted for the base case scenario of Ashayeri et al. to assess the impact of each parameter on the solution structure. This could be done for the parameter sets of TT1 to TT5 as well. As the results do not differ significantly with respect to the Ashayeri et al. base case, only a variation of the return ratio $\alpha$ for product TT1 is presented henceforth. Figure 22 compares the minimum total cost of the preset policy structures $(1, M)$, $(R, 1)$, and $(R, 1)^g$ for TT1 when $\alpha$ is altered.

In correspondence to Figure 9, the $(1, M)$ policy dominates the policies that propose to schedule more than one remanufacturing lot for small return rates. When $\alpha$ lies between 35.31% and 66.3%, a $(1, 1)$ policy is suggested by all preset policy struc-

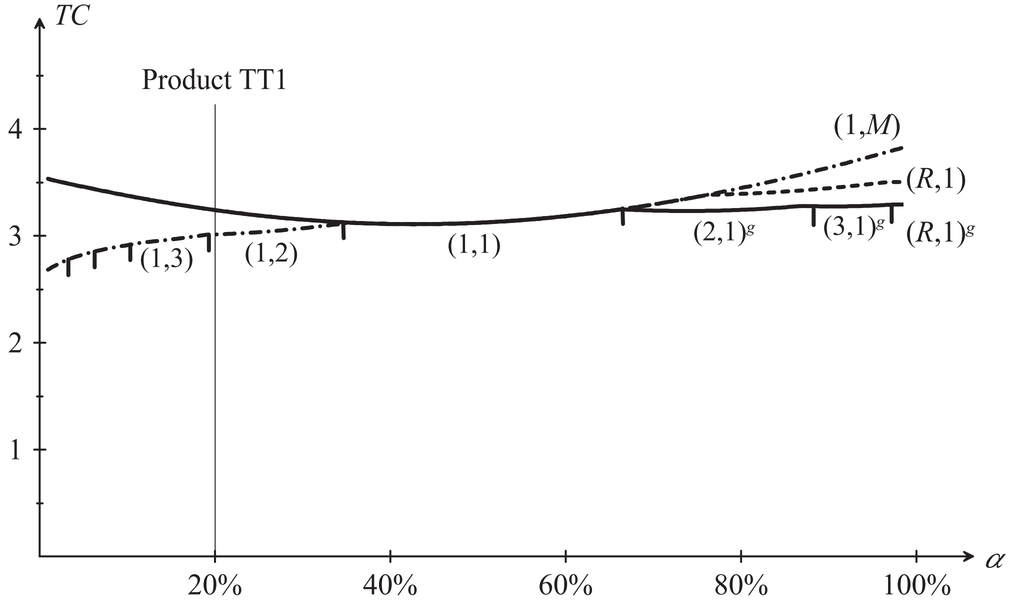


Figure 22: Minimum total cost of the preset policy structures for different $\alpha$ values (TT1)

tures. The exact values for $\alpha$ are computed, again, by evaluating equations (39) and (47). When $\alpha$ becomes larger than 66.3%, the $(R,1)^g$ policy is the best preset policy structure. Thus, Figure 23 depicts the deviation of the best preset policy structures excluding and including the $(R,1)^g$ policy from the benchmark solution. It can be observed as for the Ashayeri et al. base case that neglecting the $(R,1)^g$ policy in the decision making process results in a significant error for large return fractions. Moreover, the best preset policy structure does not deviate by more than 1.5% from the benchmark solution. The worst case deviation of 1.02% can be found when $\alpha$ amounts to 66.5%.

A similar outcome can be observed when the return ratio is altered for products TT2 to TT5. Likewise, the findings of varying the remaining parameters correspond to the Ashayeri et al. base case scenario. Therefore, we omit to present the corresponding figures and conclude this chapter by a short summary and an outlook on future research options.

# 5 Concluding remarks and outlook

After giving a short introduction to the problem setting and presenting the available literature on this topic in Section 1, two policy structures known from literature have been presented in the adjacent Section 2, Schrady's $(R,1)$ policy and Teunter's $(1,M)$ policy. Both policies rely on the assumption of equally sized batches in either the

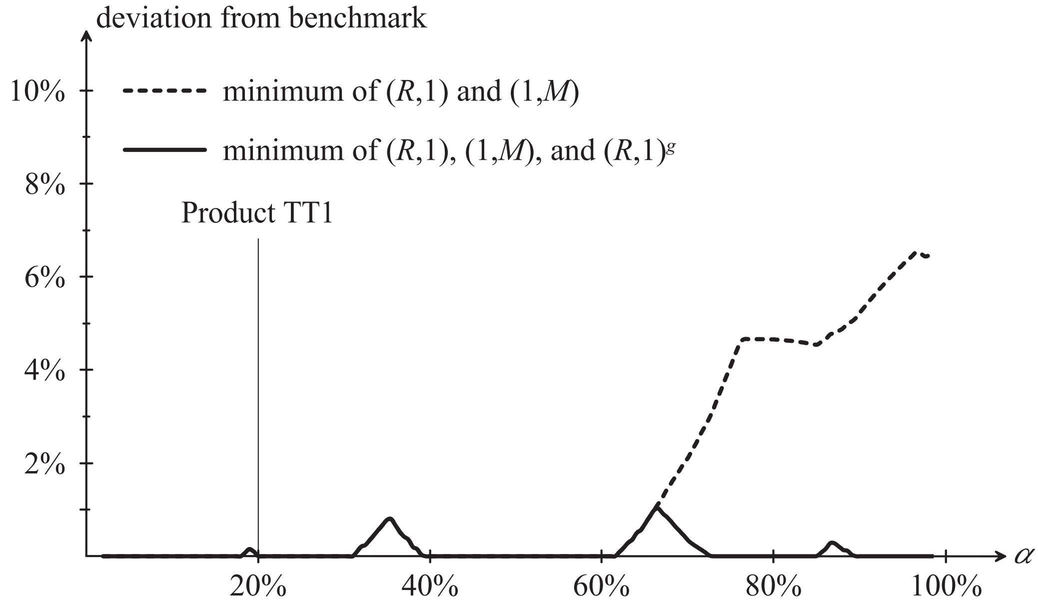


Figure 23: Deviation from benchmark solution for different $\alpha$ values (TT1)

remanufacturing (Schrady) or the manufacturing (Teunter) process and formulate the objective function depending on these lot sizing decisions. By doing this, they neglect that the number of remanufacturing lots $R$ and manufacturing lots $M$ have to be integer. Minner avoids this pitfall by reformulating both policies such that their objective function depends on the cycle length $T$ as well as on $R$ or $M$, respectively. This reformulation allows to derive closed-form expressions to determine the optimal integer $R^*$ and $M^*$ for both policy structures. Subsequently, it has been proven that it is not possible for $R^*$ and $M^*$ to be larger than one simultaneously when restricting oneself to the $(R, 1)$ and $(1, M)$ policies. This reduces the effort to compute the better of these policies. In contrast to these policy structures, a third approach has been introduced in Section 2.6, the $(R, 1)^g$ policy. Instead of initiating remanufacturing lots of equal size as the $(R, 1)$ policy requires, the $(R, 1)^g$ policy schedules geometrically decreasing remanufacturing lots. When doing this, each remanufacturing lot remanufactures all available returns which depletes the used product inventory after each remanufacturing run. Yet, a closed-form expression to generate the optimal integer value for $R$ could not be derived. However, some conditions could be determined at which this preset policy outperforms the other preset policy structures.

So far, the preset policy structures could only be compared to each other. Therefore, the main focus of the subsequent Section 3 has been to present an optimization approach in order to compute the optimal cycle. To do this, a mixed-integer non-linear problem is introduced that requires the number of remanufacturing and manufacturing batches in a cycle as input. This model has been solved to generate a benchmark solution that provides an opportunity to evaluate the performance of the three preset policy structures. This evaluation has been the subject of Section 4, the numerical study. Starting with the introduction of a base case scenario (taken from Ashayeri et al.), a sensitivity analysis is conducted that modifies each parameter individually. Several interesting aspects have been observed during this study. One of the most important aspects has been that by neglecting the $(R, 1)^g$ policy a significant error of up to 9 % could be made for some parameter combinations (in this study this was the case when the return rate $\alpha$ is large). Furthermore, the benchmark solution has in no instance been worse than the best preset policy structure. This could not be expected beforehand due to the non-linearity of the objective function and the restriction to 36 different parameter combinations of $R$ and $M$. Finally, the best solutions of the preset policies have never been worse than 1 % compared to the benchmark solution which can be interpreted as a promising result.

Several research questions remain still unanswered and can be addressed in future. At first, the preset policy structures can be extended to include policies having more than one remanufacturing and more than one manufacturing batch in a cycle simultaneously. Choi et al. (2007) have been the first authors to test this assumption. Although they restrict their analysis to general $(R, M)$ policy structures with equal remanufacturing and equal manufacturing batches, they were able to identify problem instances for which the solution can be improved. It would be interesting to evaluate the performance gain for a more general $(R, M)^g$ policy structure in which all remanufacturing batches use all available returns in stock.

Improving the benchmark solution can be another challenging task. In order to do this, the properties of the benchmark solutions have to be analyzed in greater detail to incorporate these findings into an improved optimization approach. Another opportunity would be to drop the integrality constraints of the optimization program and examine the relaxed solution to determine lower bounds for the optimal solution. It would be an interesting insight if the monotonicity in the ratio of $R^*$ and $M^*$ (as depicted in Figure 12 for a varying $\alpha$) can be confirmed when using the relaxed optimization approach instead of the original one.

In addition, several assumptions can be analyzed critically to evaluate their importance on the results presented in this chapter. In our study, remanufacturing has been considered a profitable opportunity for the OEM no matter how long the returns are kept in stock. In reality, disposing of some returns at the beginning of a cycle can be advantageous as they would have to be stored over a long time before remanufacturing. Several contributions have analyzed this setting as well as a setting with a finite production and remanufacturing rate and possible lead times. Future research efforts can examine the effect of introducing differently sized remanufacturing lots for these settings as well. Moreover, the assumption of static demand and return rates can be criticized. Incorporating time variant returns and demand can help to model a more realistic system in this context. Concluding, uncertainties can almost never be neglected in real-life systems. Uncertainties prevail for remanufacturing systems regarding their inputs as the OEM does not know how many customers return their product at what time and in which condition. Furthermore, the output is uncertain, too, since the yield of remanufacturing and the customer demand can only be estimated in advance.

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# Appendix

**Derivation of equation (4):**

$$\begin{aligned}
\frac{K_m + R \cdot K_R}{T} &= \frac{K_m + R \cdot K_R}{\frac{Q_M + R \cdot Q_R \cdot \beta}{\lambda}} \\
&= \lambda \cdot \frac{K_m + \frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot K_R}{Q_M + \frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot Q_R \cdot \beta} \\
&= \lambda \cdot \frac{K_m + \frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot K_R}{Q_M \cdot \left(1 + \frac{\alpha\beta}{1-\alpha\beta}\right)} \\
&= \lambda \cdot \frac{K_m + \frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot K_R}{\frac{Q_M}{1-\alpha\beta}} \\
&= \lambda \cdot \left(K_m \cdot \frac{1-\alpha\beta}{Q_M} + \frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot K_R \cdot \frac{1-\alpha\beta}{Q_M}\right) \\
&= \lambda \cdot \left(\frac{(1-\alpha\beta)\cdot K_M}{Q_M} + \frac{\alpha \cdot K_R}{Q_R}\right)
\end{aligned}$$

**Derivation of equation (6):**

$$\begin{aligned}
&\left(\frac{1}{2} \cdot \frac{R \cdot (Q_R \cdot \beta)^2}{\lambda} + \frac{1}{2} \cdot \frac{(Q_M)^2}{\lambda}\right) \cdot h_M \cdot \frac{1}{T} \\
= \quad & \frac{1}{2\lambda} \cdot \frac{R \cdot (Q_R \cdot \beta)^2 + (Q_M)^2}{\frac{R \cdot Q_R \cdot \beta + Q_M}{\lambda}} \cdot h_M \\
= \quad & \frac{1}{2} \cdot \frac{\frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot (Q_R \cdot \beta)^2 + (Q_M)^2}{\frac{\alpha \cdot Q_M}{(1-\alpha\beta)\cdot Q_R} \cdot Q_R \cdot \beta + Q_M} \cdot h_M \qquad \text{manipulation similar to (4)} \\
= \quad & \frac{1}{2} \cdot \frac{\frac{\alpha \cdot Q_M}{(1-\alpha\beta)} \cdot Q_R \cdot \beta^2 + (Q_M)^2}{\frac{Q_M}{1-\alpha\beta}} \cdot h_M \\
= \quad & \frac{1}{2} \cdot \left(\alpha\beta^2 \cdot Q_R + (1-\alpha\beta) \cdot Q_M\right) \cdot h_M.
\end{aligned}$$

**Proof of convexity of the total cost function in (7):** In order to prove the convexity of the total cost function $TC_{R1}$, its Hessian matrix has to be elaborated. This results in:

$$H(TC_{R1}) = \begin{bmatrix} \frac{\partial^2 TC_{R1}}{\partial (Q_R)^2} & \frac{\partial^2 TC_{R1}}{\partial Q_R \partial Q_M} \\ \frac{\partial^2 TC_{R1}}{\partial Q_M \partial Q_R} & \frac{\partial^2 TC_{R1}}{\partial (Q_M)^2} \end{bmatrix} = \begin{bmatrix} \frac{2\lambda\alpha K_R}{(Q_R)^3} & 0 \\ 0 & \frac{2\lambda(1-\alpha\beta)K_M}{(Q_M)^3} \end{bmatrix}$$

This Hessian is positive definite as its leading principal minors are strictly positive, i.e. $\frac{2\lambda\alpha K_R}{(Q_R)^3} > 0$ and $\frac{4\lambda^2 \alpha K_r K_m (1-\alpha\beta)}{(Q_R)^3 \cdot (Q_M)^3} > 0$. Therefore, the total cost function $TC_{R1}$ is jointly convex in both decision variables $Q_R$ and $Q_M$.

**Derivation of equation (15):**

$$\begin{aligned}
\frac{M \cdot K_m + K_R}{T} &= \frac{M \cdot K_m + K_R}{\frac{M \cdot Q_M + Q_R \cdot \beta}{\lambda}} \\
&= \lambda \cdot \frac{\frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot K_m + K_R}{\frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot Q_M + Q_R \cdot \beta} \\
&= \lambda \cdot \frac{\frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot K_m + K_R}{Q_R \cdot \left(\frac{1-\alpha\beta}{\alpha} + \beta\right)} \\
&= \lambda \cdot \frac{\frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot K_m + K_R}{\frac{Q_R}{\alpha}} \\
&= \lambda \cdot \left( \frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot K_m \cdot \frac{\alpha}{Q_R} + K_R \cdot \frac{\alpha}{Q_R} \right) \\
&= \lambda \cdot \left( \frac{K_M \cdot (1-\alpha\beta)}{Q_M} + \frac{K_R \cdot \alpha}{Q_R} \right)
\end{aligned}$$

**Derivation of equation (16):**

$$\begin{aligned}
&\left[ \frac{1}{2} \cdot Q_R \cdot T \cdot h_R + \left( \frac{1}{2} \cdot \frac{(Q_R \cdot \beta)^2}{\lambda} + M \cdot \frac{1}{2} \cdot \frac{(Q_M)^2}{\lambda} \right) \cdot h_M \right] \cdot \frac{1}{T} \\
= \quad & \frac{1}{2} \cdot Q_R \cdot h_R + \frac{\left( \frac{1}{2} \cdot \frac{(Q_R \cdot \beta)^2}{\lambda} + M \cdot \frac{1}{2} \cdot \frac{(Q_M)^2}{\lambda} \right)}{\frac{M \cdot Q_M + Q_R \cdot \beta}{\lambda}} \cdot h_M \\
= \quad & \frac{1}{2} \cdot Q_R \cdot h_R + \frac{1}{2} \cdot \frac{(Q_R \cdot \beta)^2 + \frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot (Q_M)^2}{\frac{Q_R \cdot (1-\alpha\beta)}{\alpha \cdot Q_M} \cdot Q_M + Q_R \cdot \beta} \cdot h_M \qquad \text{manipulation similar to (15)} \\
= \quad & \frac{1}{2} \cdot Q_R \cdot h_R + \frac{1}{2} \cdot \frac{(Q_R \cdot \beta)^2 + \frac{Q_R \cdot (1-\alpha\beta)}{\alpha} \cdot Q_M}{\frac{Q_R}{\alpha}} \cdot h_M \\
= \quad & \frac{1}{2} \cdot \left( Q_R \cdot h_R + \left( \alpha\beta^2 \cdot Q_R + (1-\alpha\beta) \cdot Q_M \right) \cdot h_M \right).
\end{aligned}$$

**Proof of convexity of the total cost function in (17):**

As for the $(R, 1)$ policy, the Hessian matrix of the total cost function has to be computed to analyze its properties. The Hessian matrix of $TC_{1M}$ is:

$$H(TC_{1M}) = \begin{bmatrix} \frac{\partial^2 TC_{1M}}{\partial (Q_R)^2} & \frac{\partial^2 TC_{1M}}{\partial Q_R \partial Q_M} \\ \frac{\partial^2 TC_{1M}}{\partial Q_M \partial Q_R} & \frac{\partial^2 TC_{1M}}{\partial (Q_M)^2} \end{bmatrix} = \begin{bmatrix} \frac{2\lambda\alpha K_R}{(Q_R)^3} & 0 \\ 0 & \frac{2\lambda(1-\alpha\beta)K_M}{(Q_M)^3} \end{bmatrix}$$

The Hessian matrix of $TC_{1M}$ coincides with the matrix for $TC_{R1}$. Thus, all leading principal minors are strictly positive again and the joint convexity (regarding $Q_R$ and

$Q_M$) of the total cost function $TC_{1M}$ is proven.

**Behavior of equation (28) near 0 and $\infty$ :**

$$TC^+_{R1}(R) = \sqrt{2\lambda(R \cdot K_R + K_M)\left(\left(1+\alpha\beta\left(\frac{1}{R}-1\right)\right)\alpha h_R + \left(\frac{\alpha^2\beta^2}{R} + (1-\alpha\beta)^2\right)h_M\right)}$$

$$TC^+_{R1}(R) = \sqrt{2\lambda\left(A \cdot R + B + \frac{C}{R}\right)}$$

$$\text{with} \quad A = K_R \cdot \left(\alpha h_R (1-\alpha\beta) + (1-\alpha\beta)^2\right) \geq 0$$

$$B = K_R \alpha^2 \beta (h_R + \beta h_M) + K_M \left(\alpha h_R (1-\alpha\beta) + (1-\alpha\beta)^2 h_M\right) \geq 0$$

$$C = K_M \alpha^2 \beta (h_R + \beta h_M)$$

$$\lim_{R\to 0} TC^+_{R1}(R) = \sqrt{2\lambda\left(A \cdot R + B + \frac{C}{R}\right)} = \infty$$

$$\lim_{R\to \infty} TC^+_{R1}(R) = \sqrt{2\lambda\left(A \cdot R + B + \frac{C}{R}\right)} = \infty$$

**Derivation of equation (29):**

$$TC^+_{R1}(\hat{R}) = TC^+_{R1}(\hat{R}+1)$$

$$\sqrt{2\lambda\left(\hat{R}K_R + K_M\right)\left(\left(1+\alpha\beta\left(\frac{1}{\hat{R}}-1\right)\right)\alpha h_R + \left(\frac{\alpha^2\beta^2}{\hat{R}} + (1-\alpha\beta)^2\right)h_M\right)}$$
$$= \sqrt{2\lambda\left(\left(\hat{R}+1\right)K_R + K_M\right)\left(\left(1+\alpha\beta\left(\frac{1}{\hat{R}+1}-1\right)\right)\alpha h_R + \left(\frac{\alpha^2\beta^2}{\hat{R}+1} + (1-\alpha\beta)^2\right)h_M\right)}$$

$$\left(\hat{R}K_R + K_M\right)\left(\left(1+\alpha\beta\left(\frac{1}{\hat{R}}-1\right)\right)\alpha h_R + \left(\frac{\alpha^2\beta^2}{\hat{R}} + (1-\alpha\beta)^2\right)h_M\right)$$
$$= \left(\left(\hat{R}+1\right)K_R + K_M\right)\left(\left(1+\alpha\beta\left(\frac{1}{\hat{R}+1}-1\right)\right)\alpha h_R + \left(\frac{\alpha^2\beta^2}{\hat{R}+1} + (1-\alpha\beta)^2\right)h_M\right)$$

$$K_M \cdot \left(\alpha^2\beta\frac{1}{\hat{R}}h_R + \alpha^2\beta^2\frac{1}{\hat{R}}h_M\right) - K_M \cdot \left(\alpha^2\beta\frac{1}{\hat{R}+1}h_R + \alpha^2\beta^2\frac{1}{\hat{R}+1}h_M\right)$$
$$-K_R \cdot \left(\alpha(1-\alpha\beta)h_R + (1-\alpha\beta)^2 h_M\right) = 0$$

$$K_M\alpha^2\beta \cdot \left(\frac{h_R}{\hat{R}} - \frac{h_R}{\hat{R}+1} + \frac{h_M\beta}{\hat{R}} - \frac{h_M\beta}{\hat{R}+1}\right) - K_R \cdot \left(\alpha(1-\alpha\beta)h_R + (1-\alpha\beta)^2 h_M\right) = 0$$

$$K_M\alpha^2\beta\cdot(h_R+h_M\beta)\cdot\left(\frac{1}{\hat{R}}-\frac{1}{\hat{R}+1}\right)-K_R\cdot\left(\alpha\left(1-\alpha\beta\right)h_R+\left(1-\alpha\beta\right)^2h_M\right)=0$$

$$\frac{\hat{R}+1-\hat{R}}{\hat{R}\cdot\left(\hat{R}+1\right)}-\frac{K_R\cdot\left(\alpha\left(1-\alpha\beta\right)h_R+\left(1-\alpha\beta\right)^2h_M\right)}{K_M\alpha^2\beta\cdot(h_R+h_M\beta)}=0$$

$$\hat{R}^2+\hat{R}-\frac{K_M\alpha^2\beta\cdot(h_R+h_M\beta)}{K_R\cdot\left(\alpha\left(1-\alpha\beta\right)h_R+\left(1-\alpha\beta\right)^2h_M\right)}=0$$

$$\hat{R}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{K_M\alpha^2\beta\cdot(h_R+h_M\beta)}{K_R\cdot\left(\alpha\left(1-\alpha\beta\right)h_R+\left(1-\alpha\beta\right)^2h_M\right)}}$$

$$R^*=\left\lceil-\frac{1}{2}\pm\sqrt{\frac{1}{4}+\frac{K_M\alpha^2\beta\cdot(h_R+h_M\beta)}{K_R\cdot\left(\alpha\left(1-\alpha\beta\right)h_R+\left(1-\alpha\beta\right)^2h_M\right)}}\ \right\rceil.$$

**Derivation of equation (36):**

$$TC_{1M}^{+}(\hat{M})=TC_{1M}^{+}(\hat{M}+1)$$

$$\sqrt{2\lambda\cdot\left(K_R+\hat{M}\cdot K_M\right)\cdot\left(\alpha h_R+\left(\alpha^2\beta^2+\frac{(1-\alpha\beta)^2}{\hat{M}}\right)\cdot h_M\right)}$$
$$=\sqrt{2\lambda\cdot\left(K_R+\left(\hat{M}+1\right)\cdot K_M\right)\cdot\left(\alpha h_R+\left(\alpha^2\beta^2+\frac{(1-\alpha\beta)^2}{\hat{M}+1}\right)\cdot h_M\right)}$$

$$\left(K_R+\hat{M}\cdot K_M\right)\cdot\left(\alpha h_R+\left(\alpha^2\beta^2+\frac{(1-\alpha\beta)^2}{\hat{M}}\right)\cdot h_M\right)$$
$$-\left(K_R+\left(\hat{M}+1\right)\cdot K_M\right)\cdot\left(\alpha h_R+\left(\alpha^2\beta^2+\frac{(1-\alpha\beta)^2}{\hat{M}+1}\right)\cdot h_M\right)=0$$

$$K_R\cdot(1-\alpha\beta)^2\cdot h_M\cdot\left(\frac{1}{\hat{M}}-\frac{1}{\hat{M}+1}\right)-K_M\cdot\left(\alpha h_R+\alpha^2\beta^2h_M\right)=0$$

$$\frac{\hat{M}+1-\hat{M}}{\hat{M}\cdot\left(\hat{M}+1\right)}-\frac{K_M\cdot(\alpha h_R+\alpha^2\beta^2h_M)}{K_R\cdot(1-\alpha\beta)^2\cdot h_M}=0$$

$$\hat{M}^2+\hat{M}-\frac{K_R\cdot(1-\alpha\beta)^2\cdot h_M}{K_M\cdot(\alpha h_R+\alpha^2\beta^2h_M)}=0$$

$$\hat{M} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{K_R \cdot (1-\alpha\beta)^2 \cdot h_M}{K_M \cdot (\alpha h_R + \alpha^2\beta^2 h_M)}}$$

$$M^* = \left\lceil -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{K_R \cdot (1-\alpha\beta)^2 \cdot h_M}{K_M \cdot (\alpha h_R + \alpha^2\beta^2 h_M)}} \right\rceil .$$

**Derivation of equation (41):**

$$\begin{aligned}
\lambda\alpha T &= Q_{R,1} + Q_{R,2} + Q_{R,3} + .. + Q_{R,R} \\
\lambda\alpha T &= Q_{R,1} + \alpha\beta Q_{R,1} + \alpha^2\beta^2 Q_{R,1} + .. + \alpha^{R-1}\beta^{R-1} Q_{R,1} \\
\lambda\alpha T &= Q_{R,1} \cdot \left(1 + \alpha\beta + \alpha^2\beta^2 + .. + \alpha^{R-1}\beta^{R-1}\right) \\
\lambda\alpha T &= Q_{R,1} \cdot \sum_{i=0}^{R-1} \alpha^i \beta^i \\
\lambda\alpha T &= Q_{R,1} \cdot \frac{1-\alpha^R\beta^R}{1-\alpha\beta} \\
Q_{R,1} &= \frac{\lambda\alpha T \cdot (1-\alpha\beta)}{1-\alpha^R\beta^R}.
\end{aligned}$$

In the transformations, the convenient formula for a geometric series has been used. In general, this formula states that $s_n = a_0 \sum_{k=0}^{n} q^k = a_0 \frac{1-q^{n+1}}{1-q}$ (for $q \neq 1$), where $a_0$ denotes the initial value and $q$ the common ratio. Here, the initial value is equal to $Q_{R,1}$ while the common ratio is $\alpha\beta$. The remaining remanufacturing batches can be calculated by using the first condition explained above. Therefore,

$$Q_{R,i} = \frac{\lambda\alpha T \cdot (1-\alpha\beta)}{1-\alpha^R\beta^R} \cdot \alpha^{i-1}\beta^{i-1} = \frac{\lambda\alpha^i\beta^{i-1} T \cdot (1-\alpha\beta)}{1-\alpha^R\beta^R} \qquad \forall i = 1, .., R.$$

**Derivation of equation (43):**

$$\begin{aligned}
&\left[\frac{1}{2}\sum_{i=1}^{R}\left(Q_{R,i} \cdot \frac{Q_{R,i}}{\lambda\alpha}\right) h_R\right] \cdot \frac{1}{T} \\
=&\left[\frac{h_R}{2\lambda\alpha}\sum_{i=1}^{R}\left(\frac{\lambda\alpha^i\beta^{i-1}T \cdot (1-\alpha\beta)}{1-\alpha^R\beta^R}\right)^2\right] \cdot \frac{1}{T} \\
=&\frac{h_R T}{2\lambda\alpha}\frac{\lambda^2 \cdot (1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2} \cdot \sum_{i=1}^{R} \alpha^{2i}\beta^{2\cdot(i-1)} \\
=&\frac{h_R T}{2\alpha}\frac{\lambda \cdot (1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2} \cdot \frac{1}{\beta^2}\sum_{i=1}^{R} \left(\alpha^2\right)^i \left(\beta^2\right)^i \\
=&\frac{h_R T}{2\alpha}\frac{\lambda \cdot (1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2} \cdot \frac{\alpha^2\beta^2}{\beta^2}\sum_{i=0}^{R-1} \left(\alpha^2\right)^i \left(\beta^2\right)^i \qquad \text{[formula for geometric series]}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{h_R T}{2}\frac{\lambda\alpha\cdot(1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2}\cdot\frac{1-\alpha^{2R}\beta^{2R}}{1-\alpha^2\beta^2}\\
=&\frac{h_R T}{2}\frac{\lambda\alpha\cdot(1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2}\cdot\frac{\left(1-\alpha^R\beta^R\right)\cdot\left(1+\alpha^R\beta^R\right)}{(1-\alpha\beta)\cdot(1+\alpha\beta)}\\
=&\frac{1}{2}\lambda\alpha T h_R\cdot\left(\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}\right).
\end{aligned}
$$

**Derivation of equation (44):**

$$
\begin{aligned}
&\left[\frac{1}{2}\left(\sum_{i=1}^{R}\left(Q_{R,i}\cdot\beta\cdot\frac{Q_{R,i}\cdot\beta}{\lambda}\right)+Q_M\cdot\frac{Q_M}{\lambda}\right)h_M\right]\frac{1}{T}\\
=&\left[\frac{h_M}{2\lambda}\left(\beta^2\sum_{i=1}^{R}\left(\frac{\lambda\alpha^i\beta^{i-1}T\cdot(1-\alpha\beta)}{1-\alpha^R\beta^R}\right)^2+\lambda^2\cdot(1-\alpha\beta)^2T^2\right)\right]\frac{1}{T}\\
=&\frac{h_M T}{2\lambda}\left(\frac{\lambda^2\cdot(1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2}\sum_{i=1}^{R}\left(\alpha^2\right)^i\left(\beta^2\right)^i+\lambda^2\cdot(1-\alpha\beta)^2\right)\\
=&\frac{\lambda h_M T}{2}\left(\frac{\alpha^2\beta^2\cdot(1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2}\sum_{i=0}^{R-1}\left(\alpha^2\right)^i\left(\beta^2\right)^i+(1-\alpha\beta)^2\right)\\
=&\frac{\lambda h_M T}{2}\left(\frac{\alpha^2\beta^2\cdot(1-\alpha\beta)^2}{(1-\alpha^R\beta^R)^2}\cdot\frac{1-\alpha^{2R}\beta^{2R}}{1-\alpha^2\beta^2}+(1-\alpha\beta)^2\right)\\
=&\frac{1}{2}\lambda T h_M\left(\alpha^2\beta^2\cdot\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^R\beta^R}{1-\alpha^R\beta^R}+(1-\alpha\beta)^2\right).
\end{aligned}
$$

**Derivation of inequality (47):**

$$TC^{+}_{R1^g}(1)-TC^{+}_{R1^g}(2)>0$$

$$
\begin{aligned}
&\sqrt{2\lambda\cdot(K_R+K_M)\cdot\left(\alpha h_R+\alpha^2\beta^2h_M+h_M\,(1-\alpha\beta)^2\right)}\\
&-\sqrt{2\lambda\cdot(2K_R+K_M)\cdot\left((\alpha h_R+\alpha^2\beta^2h_M)\cdot V+h_M\,(1-\alpha\beta)^2\right)}>0
\end{aligned}
$$

$$
\begin{aligned}
&(K_R+K_M)\cdot\left(\alpha h_R+\alpha^2\beta^2h_M+h_M\,(1-\alpha\beta)^2\right)\\
&-(2K_R+K_M)\cdot\left(\left(\alpha h_R+\alpha^2\beta^2h_M\right)\cdot V+h_M\,(1-\alpha\beta)^2\right)>0
\end{aligned}
$$

$$K_R\left(\alpha h_R+\alpha^2\beta^2h_M\right)(1-2V)-K_Rh_M(1-\alpha\beta)^2+K_M\left(\alpha h_R+\alpha^2\beta^2h_M\right)(1-V)>0.$$

**Derivation of inequality (49):**

$$\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^2\beta^2}{1-\alpha^2\beta^2})>\frac{1}{2}$$

$$
\begin{aligned}
2\left(1-\alpha\beta\right)\left(1+\alpha^2\beta^2\right) &> \left(1+\alpha\beta\right)\left(1-\alpha^2\beta^2\right)\\
2-2\alpha\beta+2\alpha^2\beta^2-2\alpha^3\beta^3 &> 1+\alpha\beta-\alpha^2\beta^2-\alpha^3\beta^3\\
1-3\alpha\beta-3\alpha^2\beta^2-\alpha^3\beta^3 &> 0\\
\left(1-\alpha\beta\right)^3 &> 0
\end{aligned}
$$

**Derivation of inequality (50):**

$$TC^{+}_{R1}(2) - TC^{+}_{R1^g}(2) > 0$$

$$\sqrt{2\lambda\cdot\left(2K_R+K_M\right)\cdot\left(\left(1+\alpha\beta\left(\frac{1}{2}-1\right)\right)\cdot\alpha h_R+\left(\frac{\alpha^2\beta^2}{2}+\left(1-\alpha\beta\right)^2\right)\cdot h_M\right)}$$
$$-\sqrt{2\lambda\cdot\left(2K_R+K_M\right)\cdot\left(\left(\alpha h_R+\alpha^2\beta^2h_M\right)\cdot V+h_M\left(1-\alpha\beta\right)^2\right)} > 0$$

$$\left(1+\alpha\beta\left(\frac{1}{2}-1\right)\right)\cdot\alpha h_R+\frac{\alpha^2\beta^2}{2}\cdot h_M-\left(\alpha h_R+\alpha^2\beta^2h_M\right)\cdot V > 0$$

$$h_M\alpha^2\beta^2V-\alpha h_R\left(-1+\frac{1}{2}\alpha\beta+V\right) > 0$$

Since V is larger than 0.5 which has been shown in condition (49), the direction of the inequality is reversed. Hence, when replacing $V$ by $\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^2\beta^2}{1-\alpha^2\beta^2}$

$$
\begin{aligned}
\frac{h_M}{h_R} &< \frac{\alpha\left(\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^2\beta^2}{1-\alpha^2\beta^2}-1+\frac{1}{2}\alpha\beta\right)}{\alpha^2\beta^2\left(\frac{1}{2}-\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^2\beta^2}{1-\alpha^2\beta^2}\right)}\\
\frac{h_M}{h_R} &< \frac{\left(\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^2\beta^2}{(1-\alpha\beta)\cdot(1+\alpha\beta)}-1+\frac{1}{2}\alpha\beta\right)}{\alpha\beta^2\left(\frac{1}{2}-\frac{1-\alpha\beta}{1+\alpha\beta}\cdot\frac{1+\alpha^2\beta^2}{(1-\alpha\beta)\cdot(1+\alpha\beta)}\right)}\\
\frac{h_M}{h_R} &< \frac{\left(\frac{1+\alpha^2\beta^2}{(1+\alpha\beta)^2}-1+\frac{1}{2}\alpha\beta\right)}{\alpha\beta^2\left(\frac{1}{2}-\frac{1+\alpha^2\beta^2}{(1+\alpha\beta)^2}\right)}\\
\frac{h_M}{h_R} &< \frac{\frac{1+\alpha^2\beta^2-(1+\alpha\beta)^2+\frac{1}{2}\alpha\beta\cdot(1+\alpha\beta)^2}{(1+\alpha\beta)^2}}{\alpha\beta^2\left(\frac{(1+\alpha\beta)^2-2\cdot(1+\alpha^2\beta^2)}{2\cdot(1+\alpha\beta)^2}\right)}\\
\frac{h_M}{h_R} &< \frac{1+\alpha^2\beta^2-1-2\alpha\beta-\alpha^2\beta^2+\frac{1}{2}\alpha\beta+\alpha^2\beta^2+\frac{1}{2}\alpha^3\beta^3}{\frac{1}{2}\alpha\beta^2\left(1+2\alpha\beta+\alpha^2\beta^2-2-2\alpha^2\beta^2\right)}\\
\frac{h_M}{h_R} &< \frac{-\frac{3}{2}\alpha\beta+\alpha^2\beta^2+\frac{1}{2}\alpha^3\beta^3}{\frac{1}{2}\alpha\beta^2\left(-1+2\alpha\beta-\alpha^2\beta^2\right)}\\
\frac{h_M}{h_R} &< \frac{-\frac{1}{2}\alpha\beta\left(3-2\alpha\beta-\alpha^2\beta^2\right)}{-\frac{1}{2}\alpha\beta^2\left(1-\alpha\beta\right)^2}\\
\frac{h_M}{h_R} &< \frac{(3+\alpha\beta)\cdot(1-\alpha\beta)}{\beta\left(1-\alpha\beta\right)^2}
\end{aligned}
$$

$$\frac{h_M}{h_R} < \frac{3+\alpha\beta}{\beta\,(1-\alpha\beta)}$$

**Basecase analysis: Determination of $\alpha$ when it is better to initiate two instead of one manufacturing lots in a cycle (page 36)**

$$K_R \cdot (1-\alpha\beta)^2 \cdot h_M - 2K_M \cdot \left(\alpha h_R + \alpha^2\beta^2 h_M\right) = 0$$

$$K_R h_M - 2K_R\alpha\beta h_M + K_R\alpha^2\beta^2 h_M - 2K_M\alpha h_R - 2K_M\alpha^2\beta^2 h_M = 0$$

$$\alpha^2\left(K_R h_M\beta^2 - 2K_M h_M\beta^2\right) - \alpha\left(2K_R h_M\beta + 2K_M h_R\right) + K_R h_M = 0$$

$$\alpha_{1,2} = \frac{K_R h_M\beta + K_M h_R}{K_R h_M\beta^2 - 2K_M h_M\beta^2} \pm \sqrt{\left(\frac{K_R h_M\beta + K_M h_R}{K_R h_M\beta^2 - 2K_M h_M\beta^2}\right)^2 - \frac{K_R h_M}{K_R h_M\beta^2 - 2K_M h_M\beta^2}}$$

$$\alpha_{1,2} = -\frac{230}{320} \pm \sqrt{\left(\frac{230}{320}\right)^2 + \frac{100}{320}}$$

$$\alpha_1 = 0.1918 \qquad \alpha_2 = -1.6291$$

As only $\alpha_1$ lies within the relevant range between 0 and 1, the value of $\alpha$ at which it is better to have only one manufacturing lot instead of two lies for the base case at 19.18 %.

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