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Supply chain coordination by contracts under binomial production yield

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Abstract

Supply chain coordination is enabled by adequately designed contracts so that decision making by multiple actors avoids efficiency losses in the supply chain. From literature it is known that in newsvendor type settings with random demand and deterministic supply the activities in supply chains can be coordinated by sophisticated contracts while the simple wholesale price contract fails to achieve coordination due to the double marginalization effect. Advanced contracts are typically characterized by risk sharing mechanisms between the actors, which have the potential to coordinate the supply chain. Regarding the opposite setting with random supply and deterministic demand, literature offers a considerably smaller spectrum of solution schemes. While contract types for the well-known stochastically proportional yield have been analyzed under different settings, other yield distributions have not received much attention in literature so far. However, practice shows that yield distributions strongly depend on the industry and the production process that is considered.

This paper analyzes a buyer-supplier supply chain in a random yield, deterministic demand setting. It is shown how under binomially distributed yields risk sharing contracts can be used to coordinate buyer's ordering and supplier's production decision. Both parties are exposed to risks of over-production and under-delivery. In contrast to settings with stochastically proportional yield, however, the impact of yield uncertainty can be quite different in the binomial yield case. Under binomial yield, the output uncertainty decreases with larger production quantities while it is independent from lot sizes under stochastically proportional yield. Consequently, the results from previous contract analyses on other yield types may not hold any longer. The current study reveals that, like under stochastically proportional yield, coordination is impeded by double marginalization if a simple wholesale price contract is applied. However, more sophisticated contracts which penalize or reward the supplier can change the risk distribution so that supply chain coordination is possible under binomial yield. Thus, even though risk diminishes with larger lot sizes, the supply chain benefits from advanced risk sharing contracts because they trigger coordinated behavior.

Key words: Supply chain coordination, contracts, binomial yield, risk sharing

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1. Introduction

Uncertainties are widely spread in supply chains with demand and supply uncertainties being the most common types. Regarding the supply side, business risks primarily result from yield uncertainty which is typical for a variety of business sectors. It frequently occurs in the agricultural sector or in the chemical, electronic and mechanical manufacturing industries (see Gurnani et al. (2000), Jones et al. (2001), Kazaz (2004), Nahmias (2009)). Here, random supply can appear due to different reasons such as weather conditions, production process risks or imperfect input material. In a supply chain context, yield or supply randomness obviously affects the risk position of the actors and, therefore, has an effect on the buyer-supplier relationship in a supply chain. The question that arises is to what extent random yields affect the decisions of the single supply chain actors and the performance of the whole supply chain. In this study we limit ourselves to a problem setting with deterministic demand. This is to focus the risk analysis of contracting on the random yield aspect which is of practical relevance for production planning in some industries (see Bassok et al. (2002)). Except for papers that address disruption risks (e.g. Asian (2014), Hou et al (2010)), all contributions in the field of contract analysis under yield randomness restrict to situations where the yield type is characterized by stochastically proportional random yields. This also holds for a prior work of Inderfurth and Clemens (2014) which considers the coordination properties of various risk sharing contracts under this type of yield randomness. In practice, however, also other yield types are found (see Yano and Lee (1995)) which need to be considered in decision making and contract analysis. A specifically important one is the type of binomially distributed yield which is observed if the defectiveness of items within a production lot is independent from unit to unit. This is found if failures in manufacturing operations or material defectives occur independently in a production process. This paper addresses the analysis of coordination by contracts under such yield conditions and investigates to which extent the results for stochastically proportional yields in Inderfurth and Clemens (2014) carry over to a situation where yields are binomially distributed.

In this context, the main purpose of this paper is to study how contracts can be used in order to diminish profit losses which are driven by uncoordinated behavior. Therefore, three different contracts are applied and analyzed regarding their coordination ability, namely the simple wholesale price contract, a reward contract (over-production risk sharing contract, first introduced by He and Zhang (2008)) and a penalty contract (compare Gurnani and Gerchak (2007)). Comparable to the newsvendor setting with stochastic demand but reliable supply, the double marginalization effect of the wholesale price contract is found in our setting. Both advanced contract types can be shown to facilitate supply chain coordination if contract parameters are chosen appropriately.

The rest of this paper is organized as follows. In section 2 the supply chain model and the yield distribution are introduced. In part 3 the centralized supply chain is analyzed and benchmark decision and profit are derived for the following contract analyses. Section 4 describes and analyzes the above mentioned contract designs with respect to their supply chain coordination potential. Section 5 summarizes main results and suggests aspects of further research.

2. Model and assumptions

This paper considers a basic single-period interaction within a serial supply chain with one buyer (indicated by B) and one supplier (indicated by S). It is assumed that all cost, price, and yield information is common knowledge. In contrast to that, deterministic end customer demand is not common knowledge but only known to the buyer. As the supplier decision is totally independent from end customer demand, this is a reasonable assumption. This setting connects to the field of contracting in a principal-agent context with information asymmetry (see Corbett and Tang (1999) or Burnetas et al. (2007)) where the principal (buyer) is better informed than the agent (supplier). Nevertheless, this property has no effect on the agent's profit because it is not a direct function of the principal's information on demand (compare Maskin and Tirole (1990)). The supply chain and the course of interaction (explained below) are depicted in Figure 1.

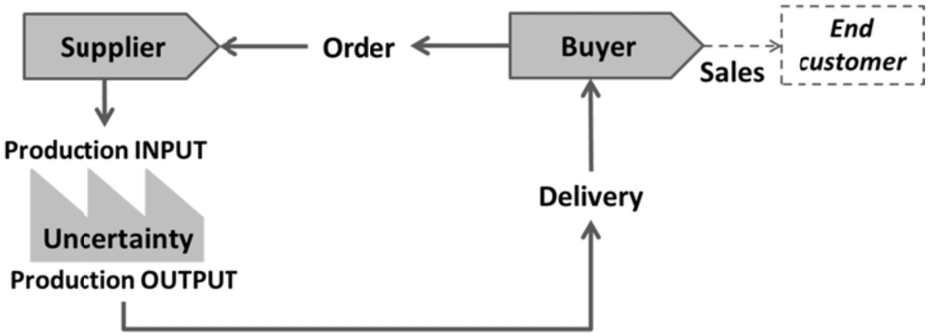


Figure 1: Serial supply chain and course of interaction

Assume the above two-member supply chain (indexed by SC). End customer demand is denoted by D. The buyer orders from the supplier an amount of X units. However, the production process of the supplier underlies risks which lead to random production yields, i.e. given identical input quantities, the amount of output in a specific production run is uncertain. The supplier can, due to production lead times, realize only a single production run.

In the following, production yield is denoted by $Y(Q)$ where Q is the production input chosen by the supplier. The quantity delivered to the buyer is the minimum of order quantity and production output. Hence, the risk of losing sales is evident to the supplier. However, it is a reasonable assumption that, given a simple wholesale price contract, the supplier is not further penalized (in addition to losing potential revenue) if end customer demand cannot be satisfied due to under-delivery. In typical business transactions the supplying side is usually measured in terms of its ability to deliver to the buyer and not to the end customer. As the mechanism to satisfy end customer demand is not in the control of the supplier, she cannot be held responsible for potential sales losses. However, both actors face the risk of lost sales because under-delivery by the supplier can cause unsatisfied demand at the buyer as stated above. Consequently, both parties may have incentives to inflate demand (from the buyer's perspective) or order quantity (from the supplier's perspective) in order to account for the yield risk and avoid lost sales. In case production output is larger than order quantity, excess units are worthless and cannot generate any revenue even though they incurred production cost. Sales at the buyer are the minimum of delivery quantity and end customer demand. If the buyer's order and delivery quantity exceed demand, excess units are also of no value and cannot be turned into revenues.

Production yields are assumed to be binomially distributed, i.e. a unit turns out 'good' (or usable) with success probability θ ($0 \le \theta \le 1$) and it is unusable with the counter probability $1-\theta$. Mean production yield under binomially distributed yield amounts to

$$\mu_{Y(Q)} = \theta \cdot Q \tag{1}$$

with a standard deviation of

$$\sigma_{Y(Q)} = \sqrt{\theta \cdot (1-\theta) \cdot Q}\,. \tag{2}$$

Note that the coefficient of variation ($\sigma_{Y(Q)} / \mu_{Y(Q)}$) decreases as the input quantity grows, i.e. the risk diminishes with increasing production quantity. This is different to the situation in Inderfurth and Clemens (2014) where production yield is a fraction of production input and neither mean nor variance of the yield rate depend on the lot size. Now, a reasonable conjecture is that under binomially distributed yields, the risk position of the single actors is different than under stochastically proportional yields. Hence, contract schemes with different risk sharing mechanisms may perform differently when the lot size influences the "amount of risk" in the supply chain and may change the proposed contract types' coordination efficiency. The subsequent analyses will shed light on this issue.

For large values of demand and the respective production quantity (i.e. if the sample of the binomial distribution is large) according to the De Moivre-Laplace theorem[2], the binomial distribution can be approximated through the Normal distribution which is done in the following.[3] This deviation from the exact binomial distribution is motivated by the fact that it facilitates the contract analysis by modeling the decision problem with continuous instead of discrete variables so that general analytic results with closed-form expressions can be derived. Furthermore, the respective numerical results are very close to optimal under fairly high demand levels. Further notation is as follows:

c	production cost [per unit]
w	wholesale price [per unit]
p	retail price [per unit]
$f_S(\cdot)$	*pdf* of standard normal distribution
$F_S(\cdot)$	*cdf* of standard normal distribution
$f_{Y(Q)}(\cdot)$	*pdf* of random variable $Y(Q)$ (yield)
$F_{Y(Q)}(\cdot)$	*cdf* of random variable $Y(Q)$ (yield)

The problem which arises is how to determine quantities for ordering on the one hand (by the buyer) and choosing a production input quantity on the other hand (by the supplier) given the risks mentioned above. The general underlying assumption in this analysis is that profitability of the business for both parties is assured, i.e. the retail price exceeds the wholesale price which in turn exceeds the expected production costs, i.e. $p > w > c/\theta$.

3. Analysis for a centralized supply chain

Under centralized decision making, the planner has only one decision to make, namely the production input quantity Q. Revenues are generated from selling to the end customer the available

[2] Compare Feller (1968) pp. 174 ff.

[3] The condition which justifies the use of the Normal distribution is the following: $Q \cdot \theta \cdot (1-\theta) > 5$ for $0.1 \leq \theta \leq 0.9$ (compare Evans et al. (2000) p. 45). In later sections, numerical examples will be conducted with success probabilities of $0.25 \leq \theta \leq 0.75$. In those cases, the Normal approximation is feasible if $Q \geq 26.67$ which holds for all our examples as shown later.

quantity, i.e. the minimum of production output and demand. Production cost, however, is incurred for every produced unit. Thus, the total supply chain profit is given by

$$\Pi_{sc}(Q) = p \cdot E\left[\min\left(D, Y(Q)\right)\right] - c \cdot Q \tag{3}$$

The first part describes the expected revenue from selling usable units; the second part constitutes the costs which are incurred by the respective production quantity. For deriving the optimal decision on production input, two cases have to be analyzed separately: $Q \le D$ and $Q \ge D$.

Case SC(I)

Under case SC(I) ($Q \le D$) it is obvious that $Y(Q) \le Q \le D$, due to $0 \le \theta \le 1$. Thus, the supply chain profit transforms to

$$\Pi_{sc}(Q) = p \cdot E\left[Y(Q)\right] - c \cdot Q = (p \cdot \theta - c) \cdot Q\,.$$

Taking the first order derivative yields

$$\frac{d\Pi_{sc}(Q)}{dQ} = p \cdot \theta - c \begin{cases} > 0 & \text{for } p > c/\theta \\ \le 0 & \text{else} \end{cases}$$

For case SC(I), it follows that the supply chain produces the following

$$Q_{SC(I)} = \begin{cases} D & \text{for } p > c/\theta \\ 0 & \text{else} \end{cases} \tag{4}$$

If the condition for profitability of the business holds, i.e. $p > c/\theta$, is has to be evaluated whether an input quantity $Q \ge D$ is preferable.

Case SC(II)

In this case ($Q \ge D$) the supply chain profit to maximize is the one in (3), namely

$$\Pi_{sc}(Q) = p \cdot E\left[\min\left(D, Y(Q)\right)\right] - c \cdot Q$$

with the sales quantity denoted by $L(D,Q) := E\left[\min\left(D, Y(Q)\right)\right] = D - \int_0^D (D-y) \cdot f_{Y(Q)}(y)\,dy$

Transforming yields[4]

$$L(D,Q) := D - \sigma_{Y(Q)} \cdot \left(F_S\left(z_{D,Q}\right) \cdot z_{D,Q} + f_S\left(z_{D,Q}\right)\right) \tag{5}$$

We define $z_{D,Q} := \dfrac{D - \mu_{Y(Q)}}{\sigma_{Y(Q)}}$ (Note that $z_{D,Q}$ depends on demand D as well as on production input Q through mean and standard deviation of the yield $Y(Q)$). Finally, the supply chain profit transforms to

$$\Pi_{sc}(Q) = p \cdot L(D,Q) - c \cdot Q \tag{6}$$

[4] For details on the transformation see Appendix A1.

Taking the first order derivative yields[5]

$$\frac{d\Pi_{SC}(Q)}{dQ} = p \cdot \frac{\partial L(D,Q)}{\partial Q} - c$$
$$= p \cdot \frac{\theta}{2} \cdot \left(2 \cdot F_S(z_{D,Q}) - \frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}} \cdot f_S(z_{D,Q}) \right) - c$$

Utilizing the first order condition $d\Pi_{SC}(Q)/dQ \overset{!}{=} 0$, the optimal input decision for case SC(II) results from the optimality condition below

$$\frac{c}{p} = \frac{\theta}{2} \cdot \left(2 \cdot F_S(z_{D,Q}) - \frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}} \cdot f_S(z_{D,Q}) \right)$$

and is denoted by $Q_{SC(II)}$. We define

$$M(D,Q) := \frac{\theta}{2} \cdot \left(2 \cdot F_S(z_{D,Q}) - \frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}} \cdot f_S(z_{D,Q}) \right) = \frac{\partial L(D,Q)}{\partial Q} \tag{7}$$

and $z_{D,Q}$ as above. Hence, the optimality condition for $Q_{SC(II)}$ can be re-formulated as

$$\frac{c}{p} = M\left(D, Q_{SC(II)}\right) \tag{8}$$

Overall solution

Since the solution from (4) is contained in (8) for $p > c/\theta$, the production decision of the supply chain as a whole is given by

$$Q^* = \begin{cases} Q_{SC(II)} & \text{for } p > c/\theta \\ 0 & \text{else} \end{cases} \tag{9}$$

The corresponding optimal profit of the supply chain results from (6) and takes the following form:

$$\Pi_{SC}^* = \Pi_{SC}(Q^*) = p \cdot D - p \cdot \left(F_S(z_{D,Q}^*) \cdot (D - \mu_{Y(Q)}^*) + \sigma_{Y(Q)}^* \cdot f_S(z_{D,Q}^*) \right) - c \cdot Q^*$$

with $\mu_{Y(Q)}^* = \mu_{Y(Q^*)}$, $\sigma_{Y(Q)}^* = \sigma_{Y(Q^*)}$, and $z_{D,Q}^* = \dfrac{D - \mu_{Y(Q)}^*}{\sigma_{Y(Q)}^*}$.

Inserting

$$\sigma_{Y(Q)}^* \cdot f_S(z_{D,Q}^*) = 2 \cdot F_S(z_{D,Q}^*) \cdot \mu_{Y(Q)}^* - \frac{2 \cdot c}{p \cdot \theta} \cdot \mu_{Y(Q)}^*$$

(which is given from (7) and (8)) into (6) yields the optimal supply chain profit:

$$\Pi_{SC}^* = p \cdot \left(1 - F_S(z_{D,Q}^*)\right) \cdot D - \left(p \cdot \theta \cdot F_S(z_{D,Q}^*) - c \right) \cdot Q^* \tag{10}$$

[5] For a detailed analysis of $\partial L(D,Q)/\partial Q$ see Appendix A2.

Concavity of the profit function is proven by showing that the second order derivative is negative as done below:[6]

$$\frac{d\Pi_{SC}(Q)^2}{d^2Q} = p\cdot\frac{\partial M(D,Q)}{\partial Q} = -f_S\left(z_{D,Q}\right)\cdot\frac{p\cdot\theta^2}{4}\cdot\frac{\left(D+\mu_{Y(Q)}+\sigma_{Y(Q)}\right)\cdot\left(D+\mu_{Y(Q)}-\sigma_{Y(Q)}\right)}{\sigma_{Y(Q)}\cdot\mu^2_{Y(Q)}} < 0 \tag{11}$$

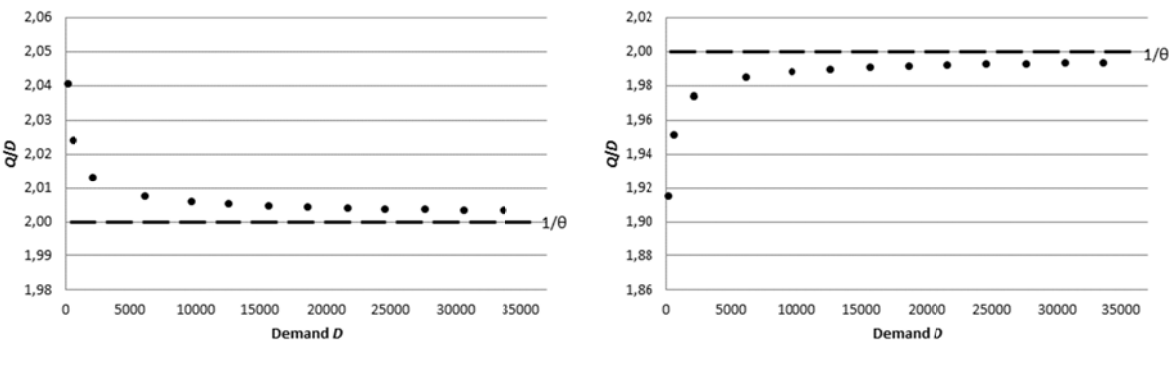
In order to analyze the relation between production quantity and demand, the derivative $dQ(D)/dD$ is evaluated. The relation between Q and D is given by[7]

$$\frac{dQ(D)}{dD} = -\frac{\partial M(D,Q)}{\partial D}\Bigg/\frac{\partial M(D,Q)}{dQ} = \frac{2\cdot\mu_{Y(Q)}\cdot\left(\mu_{Y(Q)}+D\right)}{\theta\cdot\left(\mu_{Y(Q)}+D+\sigma_{Y(Q)}\right)\left(\mu_{Y(Q)}+D-\sigma_{Y(Q)}\right)} > 0 \tag{12}$$

which shows that larger demand leads to larger production quantities which is intuitive. Interestingly, the production quantity–demand ratio (Q/D) converges to a constant the larger demand gets. Assuming that demand approaches infinity, it can be shown that the production decision approaches demand multiplied by $1/\theta$, i.e. there will be no further adjustment of demand to account for the risk. This is reasonable as binomially distributed yields decrease in risk as the input quantity rises (from $\lim_{Q\to\infty}\left(\sigma_{Y(Q)}/\mu_{Y(Q)}\right)=0$). Generally, we formulate the following Lemma:

Lemma: If demand approaches infinity, the inflation factor of demand for the production process, i.e. Q/D*, approaches* $1/\theta$*.*[8]

However, there is no distinct way how the Q/D-ratio is developing as demand grows. Rather, it depends on demand, costs, prices, and success probability whether the ratio is increasing from below $1/\theta$, decreasing from above $1/\theta$ or takes a combination of both. Thus, distinct monotony cannot be proven. Figure 2 illustrates three exemplary curves for the Q/D-ratio with increasing demand.

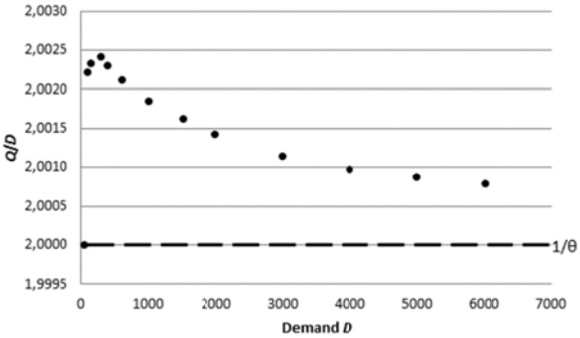


(a) Data: $c=1;\ p=6$ (b) Data: $c=1;\ p=2.5$

[6] For $\partial M(D,Q)/\partial Q$ see Appendix A3.

[7] For $\partial M(D,Q)/\partial D$ see Appendix A3.

[8] The proof of the Lemma is provided in Appendix A4.



(c) Data: $c=1$; $p=4.17$

Figure 2: Three exemplary developments for production input/demand-ratio for 50% success probability which approaches $1/\theta$

It is evident from the different curves that there is no monotony in the Q/D-ratio. Yet, the results in (a) and (b) are comparable with typical newsvendor settings where the critical ratio (here it is given by c/p) determines whether optimal production quantities are below or above expected demand (which corresponds to production yield in our setting). The major difference is that, in addition to prices and costs, also demand has an influence on the production decision as the production risk decreases with increasing quantity. A high margin (as in (a)) causes Q/D-ratios above $1/\theta$ while low margins (compare (b)) lead to production inputs below the expected yield. Yet, the shape of the curve in (c) is quite interesting. The changes in Q/D are minor with increasing demand, however, at one point the curve intersects with $1/\theta$ (which is at $D=50$). For illustrative purpose, the segment $0 \le D \le 1000$ from curve (c) is extracted in Figure 3.

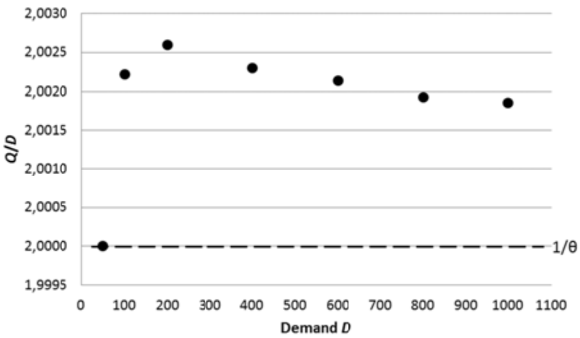
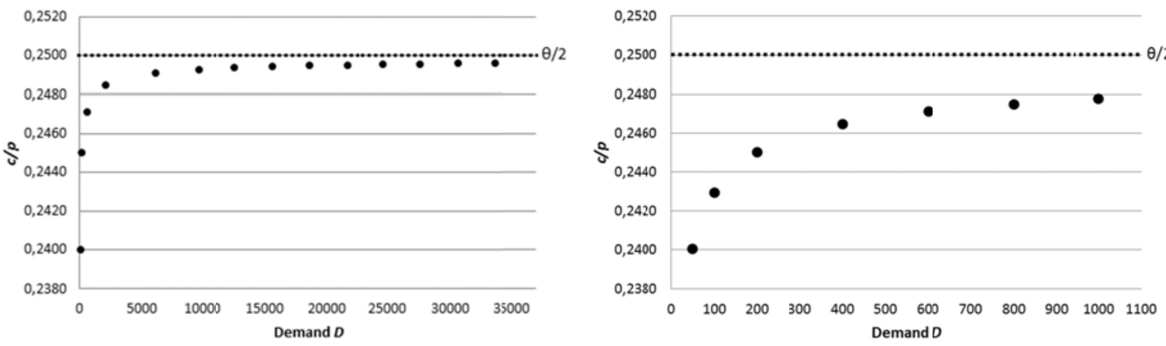


Figure 3: Extraction from Figure 2 part (c)

The intersection with $1/\theta$ raises the question whether there exist parameter combinations which always guarantee an inflation of demand in the amount of $1/\theta$. Figure 4 part (a) answers this question by illustrating the c/p ratio which results in $Q/D=1/\theta$ for increasing demand.



(a) Demand range $0 \le D \le 35000$. (b) Demand range $0 \le D \le 1000$

Figure 4: Critical parameter ratio (c/p) which guarantees a Q/D-ratio of $1/\theta$

Part (b) of the above figure extracts the range $0 \le D \le 1000$ from part (a). Comparing this illustration with Figure 3, the point $Q/D = 1/\theta$ at $D = 50$ corresponds to the starting point of the curve in Figure 4 (b) which is at $c/p = 1/4.17 = 0.24$.

4. Contract analysis for a decentralized supply chain

A decentralized supply chain consists of more than one decision maker. In our setting, a single buyer decides on the order quantity to fill end customer demand and a single supplier produces in order to satisfy the order from the buyer as described in the beginning. The decentralized supply chain is modelled as a Stackelberg game with the buyer being the leader and the supplier being the follower, i.e. the buyer anticipates the production decision by the supplier in reaction to his order.

4.1. Wholesale price contract

Under a *simple wholesale price (WHP)* contract the supplier produces the quantity Q and delivers to the buyer at a per unit wholesale price w (which is assumed to be exogenously given). The buyer's order is satisfied in the scope of the production output (at maximum). In the following, the decisions made by the supplier and by the buyer are analyzed separately.

Supplier decision

Given the buyer's order quantity X, the supplier optimizes the following profit:[9]

$$\Pi_S^{WHP}(Q|X) = w \cdot E\left[\min(X, Y(Q))\right] - c \cdot Q \tag{13}$$

The first part describes the expected revenue from selling usable units to the buyer; the second part is the corresponding production cost. Again, two cases have to be considered separately.

Case S(I)

Under case S(I) ($Q \le X$) it holds that $Y(Q) \le Q \le X$ due to $0 \le \theta \le 1$ and the supplier faces a profit of

$$\Pi_S^{WHP}(Q|X) = w \cdot E\left[Y(Q)\right] - c \cdot Q = (w \cdot \theta - c) \cdot Q \tag{14}$$

The first order derivative

[9] The following analysis is identical to the centralized case with *X* instead of *D* and *w* instead of *p*.

$$\frac{d\Pi_S^{WHP}\left(Q|X\right)}{dQ}=w\cdot\theta-c$$

is positive if $w>c/\theta$ and negative otherwise. Thus, it implies the following production decision

$$Q_{S(I)}^{WHP}\left(X\right)=\begin{cases}X & \text{for } w>c/\theta\\ 0 & \text{else}\end{cases} \tag{15}$$

If the condition for profitability of the business holds, i.e. $w>c/\theta$, it has to be evaluated whether $Q\geq X$ is preferable for the supplier.

Case S(II)

In this case ($Q\geq X$) the supplier's profit to maximize is the one in (13) and after transformation given by

$$\Pi_S^{WHP}\left(Q|X\right)=w\cdot L\left(X,Q\right)-c\cdot Q \tag{16}$$

We define the delivery quantity from the supplier to the buyer as[10]

$$L\left(X,Q\right)=X-\sigma_Y\cdot\left(F_S\left(z_{X,Q}\right)\cdot z_{X,Q}+f_S\left(z_{X,Q}\right)\right) \tag{17}$$

and $z_{X,Q}:=\frac{X-\mu_{Y(Q)}}{\sigma_{Y(Q)}}$ (Note that $z_{X,Q}$ depends on order quantity X as well as on production input Q through mean and standard deviation of the yield $Y(Q)$). Analogously, the optimal production input for case S(II) results from the first order condition below:

$$\frac{d\Pi_S^{WHP}\left(Q|X\right)}{dQ}=w\cdot\frac{\partial L\left(X,Q\right)}{\partial Q}-c\overset{!}{=}0$$

with

$$\frac{\partial L\left(X,Q\right)}{\partial Q}=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(z_{X,Q}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{X,Q}\right)\right)=M\left(X,Q\right) \tag{18}$$

which is independent from any cost or price parameter.[11] The optimal input quantity under case S(II) is denoted by $Q_{S(II)}^{WHP}$ and satisfies the optimality condition below

$$\frac{c}{w}=M\left(X,Q_{S(II)}^{WHP}\right) \tag{19}$$

Theoretically, the supplier can choose a production quantity which is smaller than the order quantity and generate positive profits. However, in this case the optimization will follow case S(I) the solution of which is included in the solution space of S(II). Summarizing, the supplier's production decision under the simple WHP contract is given by

[10] For details on the transformation of the supplier's delivery quantity, recall Appendix A1.

[11] For the first order derivative $\partial L\left(X,Q\right)/\partial Q$ recall Appendix A2.

$$Q^{WHP}(X)=\begin{cases} Q_{S(II)}^{WHP} & \text{for } w>c/\theta \\ 0 & \text{else} \end{cases} \tag{20}$$

The supplier's profit is concave as the second order derivative is negative: [12]

$$\frac{d\Pi_S^{WHP}(Q|X)^2}{d^2Q}=w\cdot\frac{\partial M(X,Q)}{\partial Q}=-f_S(z_{X,Q})\cdot\frac{w\cdot\theta^2}{4}\cdot\frac{\left(X+\mu_{Y(Q)}+\sigma_{Y(Q)}\right)\cdot\left(X+\mu_{Y(Q)}-\sigma_{Y(Q)}\right)}{\sigma_{Y(Q)}\cdot\mu_{Y(Q)}^2}<0$$

Analogously to the supply chain analysis, the relation between Q and X is given by[13]

$$\frac{dQ(X)}{dX}=-\frac{\partial M(X,Q)}{\partial X}\bigg/\frac{\partial M(X,Q)}{\partial Q}=\frac{2\cdot\mu_{Y(Q)}\cdot\left(\mu_{Y(Q)}+X\right)}{\theta\cdot\left(\mu_{Y(Q)}+X+\sigma_{Y(Q)}\right)\left(\mu_{Y(Q)}+X-\sigma_{Y(Q)}\right)}>0 \tag{21}$$

Buyer decision

The buyer as the leader in this Stackelberg game anticipates the decision made by the supplier from (20). As the first mover, the buyer's profit to maximize under a simple WHP contract is the following:

$$\Pi_B^{WHP}(X)=p\cdot E\left[\min\left(D,X,Y(Q)\right)\right]-w\cdot E\left[\min\left(X,Y(Q)\right)\right] \tag{22}$$

The first part of the profit function is the expected revenue from selling to the end customer; the second part describes the expected cost from procuring units from the supplier. In the following, it has to be evaluated whether it is preferable for the buyer to order below or above demand.

Case B(I)

Under case B(I) ($X\le D$) the buyer's profit is given by

$$\Pi_B^{WHP}(X)=(p-w)\cdot E\left[\min\left(X,Y(Q)\right)\right]=(p-w)\cdot L(X,Q) \tag{23}$$

The first order derivative is rather complex as the buyer is the leader in this Stackelberg game and accounts for the supplier's reaction to his decision, i.e. $Q=Q^{WHP}(X)$. Therefore, the total first order derivative of this function includes the relation $dQ(X)/dX$ from (21) which describes the change in production input given a change in order quantity. The total first order derivative is given by

$$\frac{d\Pi_B^{WHP}(X)}{dX}=\frac{\partial\Pi_B^{WHP}(X)}{\partial X}+\frac{\partial\Pi_B^{WHP}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX} \tag{24}$$

Given the first order derivative $\partial L(X,Q)/\partial X$ (with $L(X,Q)$ from (17)) as

$$\frac{\partial L(X,Q)}{\partial X}=1-\sigma_{Y(Q)}\cdot\left(f_S\left(z_{X,Q}\right)\cdot z_{X,Q}\cdot\frac{1}{\sigma_{Y(Q)}}+F_S\left(z_{X,Q}\right)\cdot\frac{1}{\sigma_{Y(Q)}}-f_S\left(z_{X,Q}\right)\cdot z_{X,Q}\cdot\frac{1}{\sigma_{Y(Q)}}\right)=1-F_S\left(z_{X,Q}\right) \tag{25}$$

the total first order derivative of the buyer's profit is derived below

$$\frac{\partial\Pi_B^{WHP}(X)}{\partial X}=(p-w)\cdot\frac{\partial L(X,Q)}{\partial X}=(p-w)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)$$

[12] The result is identical to (11) with *X* instead of *D* and *w* instead of *p*.

[13] The result is identical to (12) with *X* instead of *D*.

$$\frac{\partial \Pi_B^{WHP}(X)}{\partial Q} \cdot \frac{dQ(X)}{dX} = (p-w) \cdot \frac{\partial L(X,Q)}{\partial Q} \cdot \frac{dQ(X)}{dX} = (p-w) \cdot M(X,Q) \cdot \frac{dQ(X)}{dX}$$

with $\partial L(X,Q)/\partial Q$ from (18). Finally, the total first order derivative is given by

$$\frac{d\Pi_B^{WHP}(X)}{dX} = (p-w) \cdot \left(1 - F_S\left(z_{X,Q}\right)\right) + (p-w) \cdot M(X,Q) \cdot \frac{dQ(X)}{dX} \tag{26}$$

Due to $M(X,Q) > 0$, $dQ(X)/dX > 0$, and the profitability assumption $p > w$ it follows that $X^{WHP} = D$ because

$$\frac{d\Pi_B^{WHP}(X)}{dX} \begin{cases} > 0 & \text{for } p > w \\ \leq 0 & \text{else} \end{cases}$$

The order decision under case B(I) is formulated below

$$X_{B(I)}^{WHP} = \begin{cases} D & \text{for } p > w \\ 0 & \text{else} \end{cases}$$

Case B(II)

Analyzing the second case B(II) ($X \geq D$), the buyer's profit is given by

$$\Pi_B^{WHP}(X) = p \cdot E\left[\min\left(D, Y(Q)\right)\right] - w \cdot E\left[\min\left(X, Y(Q)\right)\right]$$

$$\Pi_B^{WHP}(X) = p \cdot L(D,Q) - w \cdot L(X,Q) \tag{27}$$

As under case B(I), the first order derivative is calculated by

$$\frac{d\Pi_B^{WHP}(X)}{dX} = \frac{\partial \Pi_B^{WHP}(X)}{\partial X} + \frac{\partial \Pi_B^{WHP}(X)}{\partial Q} \cdot \frac{dQ(X)}{dX}$$

As such, the single parts are given below

$$\frac{\partial \Pi_B^{WHP}(X)}{\partial X} = -w \cdot \frac{\partial L(X,Q)}{\partial X} = -w \cdot \left(1 - F_S\left(z_{X,Q}\right)\right)$$

$$\begin{aligned} \frac{\partial \Pi_B^{WHP}(X)}{\partial Q} \cdot \frac{dQ(X)}{dX} &= \left(p \cdot \frac{\partial L(D,Q)}{\partial Q} - w \cdot \frac{\partial L(X,Q)}{\partial Q}\right) \cdot \frac{dQ(X)}{dX} \\ &= \left(p \cdot M(D,Q) - w \cdot M(X,Q)\right) \cdot \frac{dQ(X)}{dX} \end{aligned}$$

with $\partial L(X,Q)/\partial X$ from (25) and $\partial L(X,Q)/\partial Q$ from (18). Finally, the total first order derivative is given by

$$\frac{d\Pi_B^{WHP}(X)}{dX} = -w \cdot \left(1 - F_S\left(z_{X,Q}\right)\right) + \left(p \cdot M(D,Q) - w \cdot M(X,Q)\right) \cdot \frac{dQ(X)}{dX} \tag{28}$$

The buyer decision under case B(II) is denoted by $X_{B(II)}^{WHP}$ and is derived from the first order condition $d\Pi_B^{WHP}(X)/dX \overset{!}{=} 0$. Hence, as the order decision under case B(II) includes the solution of case B(I), the overall order decision under the WHP contract is formulated below

$$X^{\text{WHP}} = \begin{cases} X_{B(II)}^{WHP} & \text{for } p > w \\ 0 & \text{else} \end{cases} \tag{29}$$

Interaction of buyer and supplier

In order to evaluate the coordination ability of the WHP contract it has to be analyzed whether a wholesale price value exists which induces the supplier to produce the supply chain optimal quantity. From the supply chain's and the supplier's optimality conditions in (8) and (19) we know that $\frac{c}{p} = M(D,Q^*)$ and $\frac{c}{w} = M(X,Q^{WHP})$, respectively, if $p > w > c/\theta$.

Coordination is achieved if $Q^{WHP} = Q^*$. Obviously, this is guaranteed if the following two conditions hold: (i) the buyer orders at demand level ($X^{WHP} = D$) which yields $M(X,Q^{WHP}) = M(D,Q^*)$ and (ii) the wholesale price is equal to the retail price which guarantees that $c/p = c/w$. Given $w = p$, the effect on the buyer's profit has to be evaluated. Given case B(II) ($X \geq D$), the first order derivative of the buyer profit in (28) transforms to

$$\frac{d\Pi_B^{WHP}}{dX} = -p \cdot \left(1 - F_S\left(z_{X,Q}\right)\right) + \left(p \cdot \frac{c}{p} - p \cdot \frac{c}{p}\right) \cdot \frac{dQ(X)}{dX} = -p \cdot \left(1 - F_S\left(z_{X,Q}\right)\right) < 0$$

Thus, for all values of the buyer's order in the range $X \geq D$, his marginal profit is negative. Consequently, the buyer will not order above end customer demand. Evaluating the decision spectrum $X \leq D$, the buyer profit from (23), given $w = p$, turns out to be zero:

$$\Pi_B^{WHP}(X) = (p - p) \cdot L(X, Q^{WHP}) = 0 .$$

Because the buyer's profit is zero for any order quantity below end customer demand, he is indifferent between all values from 0 to D. Assuming that the buyer orders $X^{WHP} = D$ units and given $w = p$, it follows from the supply chain's and the supplier's profits in (6) and (16) that

$$\Pi_S^{WHP}\left(Q^{WHP} \middle| X^{WHP} = D\right) = p \cdot L(D,Q) - c \cdot Q = \Pi_{SC}(Q)$$

Thus, the supplier receives the total supply chain profit while the buyer does not generate any profit when ordering D units. Hence, the buyer does not agree on the contract. Ordering zero units results in the business to not take place at all. Consequently, coordination cannot be enabled by the simple wholesale price contract if the two above conditions hold. The buyer only participates in the business if the wholesale price is below the retail price. However, in this case it holds that $c/p < c/w$ and consequently $M(X,Q^{WHP}) > M(D,Q^*)$. As $\partial M(X,Q)/\partial Q < 0$, it follows that the supplier's production quantity is too low to coordinate the supply chain. Only a wholesale price value as large as the retail price incentivizes the supplier to produce the supply chain optimal quantity. Nevertheless, a low wholesale price may induce the buyer to order larger amounts which compensate the unwillingness of the supplier to inflate the order enough to reach the supply chain optimum. For that reason, another extreme case for the wholesale price is evaluated.

If the supplier sells at her expected production cost to the buyer ($w = c/\theta$), it is obvious that a production quantity larger than the order quantity makes no sense. Thus, case S(I) $Q \le X$ must be analyzed with the profit function from (14). Setting $w = c/\theta$ yields

$$\Pi_S^{WHP}(Q) = \left(\frac{c}{\theta} \cdot \theta - c\right) \cdot Q = 0 .$$

Because the supplier's profit is zero for all possible production choices, she is indifferent between all values from 0 to X^{WHP}. That being the case, it will be assumed that the supplier produces $Q^{WHP} = X^{WHP}$ units. Anticipating this behavior, the buyer maximizes his profit for case B(II) $X \ge D$ in (27)

$$\Pi_B^{WHP}(X) = p \cdot L(D,Q) - w \cdot L(X,Q)$$

Given $Q^{WHP} = X^{WHP}$, it follows that $F_S(z_{X,Q}) = 1$ and $f_S(z_{X,Q}) = 0$.[14] Thus, the buyer's profit function transforms to

$$\Pi_B^{WHP}\left(X^{WHP} \middle| Q^{WHP} = X^{WHP}\right) = p \cdot L(D,Q) - c \cdot Q = \Pi_{SC}(Q)$$

because according to (5) $w \cdot L(X,Q) = \frac{c}{\theta} \cdot L(X,Q) = \frac{c}{\theta} \cdot Q + \frac{c}{\theta} \cdot \left(1 \cdot (Q - \theta \cdot Q) + \sigma_{Y(Q)} \cdot 0\right) = c \cdot Q$.

As $X^{WHP} = Q^{WHP}$ and $\Pi_B^{WHP}\left(X^{WHP} \middle| Q^{WHP} = X^{WHP}\right) = \Pi_{SC}(Q)$, obviously it follows that $X^{WHP} = Q^*$ and $\Pi_B^{WHP}\left(X^{WHP}\right) = \Pi_{SC}\left(Q^*\right)$.

It can be shown that given $w = c/\theta$, coordination of the supply chain will be enabled with the buyer ordering the supply chain optimal production quantity and the supplier producing the exact order quantity. However, as the supplier is left with no profit, her participation constraint is violated and she does not agree on the contract. Thus, coordination of the supply chain is impeded by violating the supplier's participation constraint.

Summarizing, each case violates the participation constraint of one actor in the supply chain ($\Pi_B^{WHP}(X) = 0$ for $w = p$ and $\Pi_S^{WHP}(Q|X) = 0$ for $w = c/\theta$) and, thus, terminates the interaction. The numerical examples in the following section provide insight that apart from the extreme values, no wholesale price in the range $c/\theta < w < p$ can enable supply chain coordination. Thus, it is concluded that the simple WHP contract fails to coordinate the supply chain.

Numerical examples

The insights become more evident when the results of the preceding analysis are demonstrated by means of a numerical example. We set the parameters as follows: $c = 1$, $p = 14$ and $D = 100$. The binomially distributed yield is approximated by the normal distribution with mean and standard

[14] For the proof see Appendix A5.

deviation from (1) and (2). For $Q \geq D = 100$ this approximation is feasible for $0.06 \leq \theta \leq 0.94$ because for these values the condition $Q \cdot \theta \cdot (1-\theta) > 5$ is satisfied.[15]

In this numerical example, we calculate the impact of different values of the wholesale price (in the interval $c/\theta \leq w \leq p$) on the profit split in the supply chain for different values of success probability θ. The benchmarks (supply chain optimal decision and profit) are also given in each table.

w	Q^*	Q^{WHP}	X^{WHP}	Π_S^{WHP}	Π_B^{WHP}	$\Pi_S^{WHP} + \Pi_B^{WHP}$	Π_{SC}^*	Q^{WHP}/D
4	419	419	419	0	958	958	958	4,19
5	419	412	111	99	858	957	958	4,12
6	419	407	106	193	763	955	958	4,07
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	419	413	100	762	196	957	958	4,13
13	419	416	100	860	98	958	958	4,16
14	419	419	100	958	0	958	958	4,19

Table 1: Effect of wholesale price on profit distribution for 25% success probability

w	Q^*	Q^{WHP}	X^{WHP}	Π_S^{WHP}	Π_B^{WHP}	$\Pi_S^{WHP} + \Pi_B^{WHP}$	Π_{SC}^*	Q^{WHP}/D
2	215	215	215	0	1177	1177	1177	2,15
3	215	211	109	101	1075	1176	1177	2,11
4	215	207	104	196	977	1173	1177	2,07
5	215	205	101	289	881	1170	1177	2,05
6	215	205	100	384	787	1171	1177	2,05
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	215	213	100	978	199	1177	1177	2,13
13	215	214	100	1078	99	1177	1177	2,14
14	215	215	100	1177	0	1177	1177	2,15

Table 2: Effect of wholesale price on profit distribution for 50% success probability

[15] Compare chapter 3 (*Analysis for a centralized supply chain*).

w	Q^*	Q^{WHP}	X^{WHP}	Π_S^{WHP}	Π_B^{WHP}	$\Pi_S^{WHP}+\Pi_B^{WHP}$	Π_{SC}^*	Q^{WHP}/D
1,33	142	142	142	0	1254	1254	1254	1,42
2	142	141	108	68	1186	1254	1254	1,41
3	142	138	103	166	1086	1252	1254	1,38
4	142	137	101	262	989	1251	1254	1,37
5	142	137	100	358	892	1250	1254	1,37
6	142	138	100	458	794	1252	1254	1,38
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
13	142	142	100	1154	100	1254	1254	1,42
14	142	142	100	1254	0	1254	1254	1,42

Table 3: Effect of wholesale price on profit distribution for 75% success probability

From all tables the interplay of production and order sizes for different wholesale price levels becomes visible, and it is illustrated how the supply chain loses efficiency if the supply chain internal price deviates from both its minimum and maximum feasible levels.[16]

4.2. Over-production risk sharing contract

Under the *over-production risk sharing (ORS)* contract the risk of producing too many units (i.e. those units which exceed the order quantity) is shared among the two parties. Thus, the supplier bears less risk and is motivated to respond to the buyer's order with a higher production quantity. Under this contract, the buyer commits to pay for all units produced by the supplier. While he pays the wholesale price w per unit for deliveries up to his actual order volume, quantities that exceed this amount are compensated at a lower price w_O. In order to exclude situations where the supplier will generate unlimited profits from over-production the following parameter restrictions are set: $w_O < c/\theta < w$. As the supplier is able to generate revenue for every produced unit she has an incentive to produce a larger lot compared to the situation under the simple WHP contract. This increase might provide the potential to align the supplier's production decision with the supply chain optimal one.

In this context two contract variants have to be distinguished depending on the way a possible over-production is handled by the parties. Under the first variant the buyer just financially compensates the supplier for over-production without physically receiving deliveries that exceed his order size. This Pull-ORS contract leaves him in a different risk position as when the parties agree that the

[16] None of the decisions outside the limiting values of the wholesale price coordinate the supply chain. Apparently optimal decisions and profits result from rounding and can be shown to be suboptimal in decimal places.

supplier will deliver the whole production output irrespective of the buyer's order. This variant is denoted a Push-ORS contract.

Supplier decision

The profit to optimize by the supplier is identical for both contract variants. It now also includes the compensation for over-production and is given by

$$\Pi_S^{ORS}\left(Q|X\right)=w\cdot E\left[\min\left(X,Y(Q)\right)\right]+w_O\cdot E\left[\left(Y(Q)-X\right)^+\right]-c\cdot Q \tag{30}$$

In the following, two cases are analyzed separately, S(I) ($Q\le D$) and S(II) ($Q\ge D$).

Case S(I)

From case S(I) ($Q\le D$) it results that $Y(Q)\le Q\le D$ and the supplier's profit transforms to

$$\Pi_S^{ORS}\left(Q|X\right)=w\cdot E\left[Y(Q)\right]+w_O\cdot 0-c\cdot Q=\left(w\cdot\theta-c\right)\cdot Q \tag{31}$$

For the first order derivative it holds that

$$\frac{d\Pi_S^{ORS}\left(Q|X\right)}{dQ}=w\cdot\theta-c\begin{cases}>0 & \text{for } w>c/\theta\\ \le 0 & \text{else}\end{cases}$$

From that, the optimal input decision under case S(I) is given by

$$Q_{S(I)}^{ORS}\left(X\right)=\begin{cases}X & \text{for } w>c/\theta\\ 0 & \text{else}\end{cases} \tag{32}$$

Consequently, it has to be evaluated whether case S(II) ($Q\ge D$) is preferable for the supplier.

Case S(II)

In this case, the supplier profit is given by

$$\begin{aligned}\Pi_S^{ORS}\left(Q|X\right)&=w\cdot E\left[\min\left(X,Y(Q)\right)\right]+w_O\cdot E\left[Y(Q)-\min\left(X,Y(Q)\right)\right]-c\cdot Q\\&=\left(w-w_O\right)\cdot E\left[\min\left(X,Y(Q)\right)\right]+w_O\cdot E\left[Y(Q)\right]-c\cdot Q\end{aligned}$$

$$\Pi_S^{ORS}\left(Q|X\right)=\left(w-w_O\right)\cdot L\left(X,Q\right)+w_O\cdot\mu_{Y(Q)}-c\cdot Q \tag{33}$$

with $L\left(X,Q\right)$ from (17). The first order derivative of the supplier's profit is given by

$$\frac{d\Pi_S^{ORS}\left(Q|X\right)}{dQ}=\left(w-w_O\right)\cdot\frac{\partial L\left(X,Q\right)}{\partial Q}+w_O\cdot\theta-c=\left(w-w_O\right)\cdot M\left(X,Q\right)+w_O\cdot\theta-c \tag{34}$$

with $\partial L\left(X,Q\right)/\partial Q$ from (18). It results the optimality condition for the supplier's production quantity under case S(II), $Q_{S(II)}^{ORS}$, from the first order condition $d\Pi_S^{ORS}\left(Q|X\right)/dQ\overset{!}{=}0$:

$$\frac{c-w_O\cdot\theta}{w-w_O}=M\left(X,Q_{S(II)}^{ORS}\right) \tag{35}$$

Thus, the supplier's production decision under an ORS contract can be formulated as

$$Q^{ORS}(X)=\begin{cases} Q_{S(II)}^{ORS} & \text{if } w>c/\theta \\ 0 & \text{else} \end{cases} \tag{36}$$

Note that for $w_O=0$ the optimal decision is identical to that under a WHP contract.

The supplier's profit is concave as the second order derivative is negative: [17]

$$\frac{d\Pi_S^{ORS}(Q|X)^2}{d^2Q}=(w-w_O)\cdot\frac{\partial M(X,Q)}{\partial Q}=-f_S(z_{X,Q})\cdot\frac{(w-w_O)\cdot\theta^2}{4}\cdot\frac{(X+\mu_{Y(Q)}+\sigma_{Y(Q)})\cdot(X+\mu_{Y(Q)}-\sigma_{Y(Q)})}{\sigma_{Y(Q)}\cdot\mu_{Y(Q)}^2}<0$$

Since $M(X,Q)$ in (35) is a constant like for the WHP contract, the first-order derivative $dQ^{ORS}(X)/dX$ is identical to that in (21).

Buyer decision

The buyer's profit function depends on the specific type of ORS contract that is applied. Under a *Pull-ORS type* (exclusion of over-delivery) the buyer maximizes a profit which compared to the WHP contract is reduced by the supplier's compensation for over-produced items

$$\Pi_B^{ORS}(X)=p\cdot E\left[\min(D,X,Y(Q))\right]-w\cdot E\left[\min(X,Y(Q))\right]-w_O\cdot E\left[(Y(Q)-X)^+\right] \tag{37}$$

As for the supplier, they buyer analysis considers two separate cases.

Case B(I)

Under case B(I) ($X\leq D$), the buyer's profit is given by

$$\begin{aligned}\Pi_B^{ORS}(X)&=(p-w)\cdot E\left[\min(X,Y(Q))\right]-w_O\cdot E\left[(Y(Q)-X)^+\right]\\&=(p-w+w_O)\cdot E\left[\min(X,Y(Q))\right]-w_O\cdot E\left[Y(Q)\right]\end{aligned}$$

$$\Pi_B^{ORS}(X)=(p-w+w_O)\cdot L(X,Q)-w_O\cdot\mu_{Y(Q)} \tag{38}$$

with $L(X,Q)$ from (17). The total first order derivative of (38) is given by[18]

$$\frac{d\Pi_B^{ORS}(X)}{dX}=(p-w+w_O)\cdot(1-F_S(z_{X,Q}))+((p-w+w_O)\cdot M(X,Q)-w_O\cdot\theta)\cdot\frac{dQ(X)}{dX} \tag{39}$$

with $M(X,Q)$ from (18) and $dQ(X)/dX$ from (21). Depending on whether the first order derivative is positive or negative, the order quantity under case B(I), $X_{B(I)}^{ORS}$, ranges from zero up to demand *D*.

Case B(II)

For case B(II) ($X\geq D$) the buyer maximizes the following profit

[17] For $\partial M(X,Q)/\partial Q$ see Appendix A3.

[18] A detailed analysis of the first order derivative is given in Appendix 6.1.

$$\Pi_B^{ORS}(X) = p \cdot E\left[\min\left(D, Y(Q)\right)\right] - (w - w_O) \cdot E\left[\min\left(X, Y(Q)\right)\right] - w_O \cdot E\left[Y(Q)\right]$$

$$\Pi_B^{ORS}(X) = p \cdot L(D,Q) - (w - w_O) \cdot L(X,Q) - w_O \cdot \mu_{Y(Q)} \tag{40}$$

with $L(D,Q)$ from (5) and $L(X,Q)$ from (17). The profit maximizing order quantity for case B(II), $X_{B(II)}^{ORS}$, results from the first order derivative below[19]

$$\frac{d\Pi_B^{ORS}(X)}{dX} = -(w - w_O) \cdot \left(1 - F_S\left(z_{X,Q}\right)\right) + \left(p \cdot M(D,Q) - (w - w_O) \cdot M(X,Q) - w_O \cdot \theta\right) \cdot \frac{dQ(X)}{dX} \tag{41}$$

with $M(D,Q)$ and $M(X,Q)$ from (7) and (18), respectively, by setting $d\Pi_B^{ORS}(X)/dX \overset{!}{=} 0$.

Interaction of buyer and supplier

As under the WHP contract, it has to be analyzed whether there exists a combination of contract parameters which guarantees that the total supply chain profit is maximized while both, supplier and buyer accept the contract. Coordination is achieved if the optimality conditions of supply chain and supplier under an ORS contract are identical. They are given from (8) and (35), respectively:

$\frac{c}{p} = M\left(D, Q^*\right)$ and $\frac{c - w_O \cdot \theta}{w - w_O} = M\left(X, Q^{ORS}\right)$.

This condition is fulfilled if (i) the buyer orders at demand level, i.e. if $X^{ORS} = D$ and (ii) if $M\left(D, Q^*\right) = M\left(X, Q^{ORS}\right)$ holds, i.e. if the following condition for the contract parameters is satisfied

$$c \cdot (w - w_O) = p \cdot (c - w_O \cdot \theta) \tag{42}$$

which ensures that $c/p = (c - w_O \cdot \theta)/(w - w_O)$. This condition also implies that $p = (w - w_O) \cdot c/(c - w_O \cdot \theta) > w - w_O$.

Given the condition for setting the parameters, the supplier's marginal profit under case S(II) in (34) turns out to be

$$\frac{d\Pi_S^{ORS}\left(Q^{ORS} = Q^* \middle| X^{ORS} = D\right)}{dQ} = (w - w_O) \cdot \frac{(c - w_O \cdot \theta)}{(w - w_O)} + w_O \cdot \theta - c = 0.$$

The supplier's marginal profit being zero, shows that the supplier actually chooses the respective quantity. As the buyer anticipates this behavior, it can be evaluated which order decision maximizes the buyer's profit. Under case B(II) ($X \geq D$), for $Q^{ORS} = Q^*$ the buyer's marginal profit from (41) transforms to

$$\begin{aligned}\frac{d\Pi_B^{ORS}(X)}{dX} &= -(w - w_O) \cdot \left[1 - F_S(z_{D,Q})\right] + \left(p \cdot \frac{c}{p} - (w - w_O) \cdot \left(\frac{c - w_O \cdot \theta}{(w - w_O)}\right) - w_O \cdot \theta\right) \cdot \frac{dQ(X)}{dX} \\ &= -(w - w_O) \cdot \left[1 - F_S(z_{D,Q})\right] + (c - c) \cdot \frac{dQ(X)}{dX} = -(w - w_O) \cdot \left[1 - F_S(z_{D,Q})\right] < 0\end{aligned}$$

[19] For more details on the derivative, see Appendix 6.1.

Due to the first-order derivative being negative, the buyer will not order above demand. Assuming an order quantity of $X^{ORS}=D$ and the coordinating parameter setting from (42), the buyer maximizes the profit under case B(I) ($X \leq D$) in (38) according to

$$\Pi_B^{ORS}\left(X^{ORS}=D\right)=\left(p-w+w_O\right)\cdot L\left(D,Q^*\right)-w_O\cdot\mu_{Y(Q)}^* .$$

Rearranging the above profit yields:

$$\begin{aligned}\Pi_B^{ORS}\left(X^{ORS}=D\right)&=p\cdot L\left(D,Q^*\right)-c\cdot Q^*+c\cdot Q^*-\left(w-w_O\right)\cdot L\left(D,Q^*\right)-w_O\cdot\theta\cdot Q^*\\&=\Pi_{SC}^*-\left(w-w_O\right)\cdot L\left(D,Q^*\right)+\left(c-w_O\cdot\theta\right)\cdot Q^*\\&=\Pi_{SC}^*-\left(w-w_O\right)\cdot L\left(D,Q^*\right)+\frac{c}{p}\cdot\left(w+w_O\right)\cdot Q^*=\Pi_{SC}^*-\left(w-w_O\right)\cdot\frac{\Pi_{SC}^*}{p}\end{aligned}$$

$$\Pi_B^{ORS}\left(X^{ORS}=D\right)=\Pi_{SC}^*\cdot\left(1-\frac{w-w_O}{p}\right) \tag{43}$$

Due to (42) it holds that $p>w-w_O$ and thus, $\Pi_B^{ORS}\left(X^{ORS}=D\right)>0$. Utilizing the first order condition of the above profit, the optimal order quantity is determined. The relation in (43) allows us to conclude that $d\Pi_B^{ORS}(X)/dX>0$ if $d\Pi_{SC}^*(X)/dX>0$ (with $\Pi_{SC}^*(X)=\Pi_{SC}^*$ for $D=X$) and thus, $X^{ORS}=D$:[20]

$$\frac{d\Pi_{SC}^*(X)}{dX}=p\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\left(p\cdot M(X,Q)-c\right)\cdot\frac{dQ(X)}{dX} \tag{44}$$

Given $M\left(D,Q^*\right)=M\left(X,Q^{ORS}\right)$, it follows that

$$\frac{d\Pi_{SC}^*(X)}{dX}=p\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\left(p\cdot\frac{c}{p}-c\right)\cdot\frac{dQ(X)}{dX}=p\cdot\left(1-F_S\left(z_{X,Q}\right)\right)>0$$

Due to $d\Pi_{SC}^*(X)/dX>0$, it is inferred that $d\Pi_B^{ORS}(X)/dX>0$ and the buyer actually orders at demand level. So, both conditions for coordination are fulfilled which proves that the *Pull-ORS* contract can enable supply chain coordination, because the buyer incentives the supplier to produce the supply chain optimal amount by ordering at demand level if the contract parameters are fixed appropriately.

If the actors agree on a *Push-ORS* contract the situation changes. In case all produced items are physically delivered, the buyer's sales are not restricted by his own order and his profit turns out to be identical for the cases B(I) and B(II), i.e. for $X \leq D$ and $X \geq D$, and is given from (40):

$$\Pi_B^{ORS}(X)=p\cdot L(D,Q)-\left(w-w_O\right)\cdot L(X,Q)-w_O\cdot\mu_{Y(Q)} .$$

From the previous analysis of the interaction between supplier and buyer, it is given that coordination requests $X^{ORS}=D$ and $c\cdot\left(w-w_O\right)=p\cdot\left(c-w_O\cdot\theta\right)$. These conditions result in the following marginal profit for the buyer:

[20] For more details on the first order derivative, see Appendix 6.1.

$$\frac{d\Pi_B^{ORS}(X)}{dX} = -(w-w_O)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\left(p\cdot\frac{c}{p}-(w-w_O)\cdot\frac{c-w_O\cdot\theta}{w-w_O}-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$
$$= -(w-w_O)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)<0$$

As the buyer's marginal profit is negative (given $w_O < w$), it is no option for the buyer to order at demand level. Through the design of the contract, orders below demand may be optimal. As the delivered quantity can exceed the order or even end customer demand, the buyer can still meet demand by 'under-ordering'. Assuming the buyer orders below demand, there may be combinations of w and w_O which incentivize the supplier to produce the supply chain optimal quantity (obviously, a larger wholesale price or a higher compensation for over-stock is necessary). However, higher prices are less profitable for the buyer who would further reduce his order quantity. This downward trend continues until nothing is ordered at all. Yet, the *Push-ORS* contract cannot coordinate the supply chain.

Numerical examples

The data for the numerical study were introduced in the section of the WHP contract. In order to fulfil the optimality condition for the contract parameters in (42), the following has to hold:

$$w_O = \frac{c\cdot(p-w)}{p\cdot\theta-c} = \frac{14-w}{14\cdot\theta-1}.$$

The tables below show the impact of the changing contract parameters on the split of profits between the buyer and the supplier in the supply chain for various success probabilities θ. Again, a normal approximation to the binomial distribution is applied. All examples show that the condition $Q\cdot\theta\cdot(1-\theta)>5$ is satisfied as production quantities are larger than 100.[21]

w	w_O	$Q^{ORS}=Q^*$	$X^{ORS}=D$	Π_S^{ORS}	Π_B^{ORS}	$\Pi_S^{ORS}+\Pi_B^{ORS}=\Pi_{SC}^*$	Q^{ORS}/D
4	4	419	100	0	958	958	4,19
5	3,6	419	100	96	862	958	4,19
6	3,2	419	100	192	766	958	4,19
7	2,8	419	100	287	671	958	4,19
8	2,4	419	100	383	575	958	4,19
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	0,8	419	100	766	192	958	4,19
13	0,4	419	100	862	96	958	4,19
14	0	419	100	958	0	958	4,19

Table 4: Effect of changing values for w and w_O on profit distribution for 25% success probability

[21] Recall chapter 3 of the paper (*Analysis for a centralized supply chain*)

w	w_O	$Q^{ORS} = Q^*$	$X^{ORS} = D$	Π_S^{ORS}	Π_B^{ORS}	$\Pi_S^{ORS} + \Pi_B^{ORS} = \Pi_{SC}^*$	Q^{ORS}/D
2	2	215	100	0	1177	1177	2,15
3	1,83	215	100	98	1079	1177	2,15
4	1,67	215	100	196	981	1177	2,15
5	1,50	215	100	294	883	1177	2,15
6	1,33	215	100	392	785	1177	2,15
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	0,33	215	100	981	196	1177	2,15
13	0,17	215	100	1079	98	1177	2,15
14	0	215	100	1177	0	1177	2,15

Table 5: Effect of changing values for w and w_O on profit distribution for 50% success probability

w	w_O	$Q^{ORS} = Q^*$	$X^{ORS} = D$	Π_S^{ORS}	Π_B^{ORS}	$\Pi_S^{ORS} + \Pi_B^{ORS} = \Pi_{SC}^*$	Q^{ORS}/D
1,33	1,33	142	100	0	1254	1254	1,42
2	1,26	142	100	66	1188	1254	1,42
3	1,16	142	100	165	1089	1254	1,42
4	1,05	142	100	264	990	1254	1,42
5	0,95	142	100	363	891	1254	1,42
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	0,21	142	100	1056	198	1254	1,42
13	0,11	142	100	1155	99	1254	1,42
14	0	142	100	1254	0	1254	1,42

Table 6: Effect of changing values for w and w_O on profit distribution for 75% success probability

4.3. Penalty contract

If a *penalty* (PEN) contract is applied the supplier will bear a higher risk than under a simple WHP contract since she will be punished for under-delivery. The supplier is penalized by the buyer (in the amount of π) for each unit ordered that cannot be delivered because of insufficient production yield. Given the potential penalty the supplier has an incentive to produce more than under the simple WHP contract which might be sufficient to achieve coordination of the supply chain.

Supplier decision

Under the PEN contract, the profit to optimize by the supplier includes the wholesale price as well as a penalty for under-delivery and is given by

$$\Pi_S^{PEN}\left(Q|X\right)=w\cdot E\left[\min\left(X,Y(Q)\right)\right]-\pi\cdot E\left[\left(X-Y(Q)\right)^{+}\right]-c\cdot Q \tag{45}$$

In the following, the two cases S(I) ($Q\leq X$) and S(II) ($Q\geq X$) are, again, analyzed separately.

Case S(I)

Given case S(I) ($Q\leq X$) the supplier's profit simplifies to

$$\Pi_S^{PEN}\left(Q|X\right)=w\cdot E\left[Y(Q)\right]-\pi\cdot\left(X-E\left[Y(Q)\right]\right)-c\cdot Q=\left((w+\pi)\cdot\theta-c\right)\cdot Q-\pi\cdot X \tag{46}$$

From the first order derivative of (46) which is given by

$$\frac{d\Pi_S^{PEN}\left(Q|X\right)}{dQ}=(w+\pi)\cdot\theta-c$$

it follows that the supplier produces either zero or the ordered amount depending on the parameter constellation as formulated below

$$\frac{d\Pi_S^{PEN}\left(Q|X\right)}{dQ}\begin{cases}>0 & \text{for } w+\pi>\dfrac{c+\pi}{\theta}\\ \leq 0 & \text{else}\end{cases}.$$

Note that if $Q=X$, then $\Pi_S^{PEN}\left(Q|X\right)=\left((w+\pi)\cdot\theta-c-\pi\right)\cdot X$ which constitutes the parameter condition above. Finally, the production quantity under case S(I), $Q_{S(I)}^{PEN}$, is formulated as follows

$$Q_{S(I)}^{PEN}(X)=\begin{cases}X & \text{for } w+\pi>\dfrac{c+\pi}{\theta}\\ 0 & \text{else}\end{cases} \tag{47}$$

Case S(II)

Assuming that $w+\pi>(c+\pi)/\theta$ holds, case S(II) ($Q\geq X$) has to be evaluated. The profit generated by the supplier is according to (45)

$$\Pi_S^{PEN}\left(Q|X\right)=w\cdot E\left[\min\left(X,Y(Q)\right)\right]-\pi\cdot E\left[X-\min\left(X,Y(Q)\right)\right]-c\cdot Q$$

$$\Pi_S^{PEN}\left(Q|X\right)=(w+\pi)\cdot L(X,Q)-\pi\cdot X-c\cdot Q \tag{48}$$

with $L(X,Q)$ from (17). Taking the first order derivative yields

$$\frac{d\Pi_S^{PEN}\left(Q|X\right)}{dQ}=(w+\pi)\cdot\frac{\partial L(X,Q)}{\partial Q}-c=(w+\pi)\cdot M(X,Q)-c \tag{49}$$

with $\partial L(X,Q)/\partial Q$ from (18). Hence, from $d\Pi_S^{PEN}(Q|X)/dQ \overset{!}{=} 0$ the optimal production input under case S(II), $Q_{S(II)}^{PEN}$, satisfies the following equation

$$\frac{c}{w+\pi} = M\left(X, Q_{S(II)}^{PEN}\right) \tag{50}$$

Hence, the supplier's production policy under a PEN contract is the following

$$Q^{PEN}(X) = \begin{cases} Q_{S(II)}^{PEN} & \text{for } w+\pi > \dfrac{c+\pi}{\theta} \\ 0 & \text{else} \end{cases} \tag{51}$$

Note that for $\pi = 0$ the optimal decision is identical to that under a WHP contract.

The supplier's profit is concave as the second order derivative is negative:[22]

$$\frac{d\Pi_S^{PEN}(Q|X)^2}{d^2Q} = (w+\pi)\cdot\frac{\partial M(X,Q)}{\partial Q} = -f_S\left(z_{X,Q}\right)\cdot\frac{(w+\pi)\cdot\theta^2}{4}\cdot\frac{\left(X+\mu_{Y(Q)}+\sigma_{Y(Q)}\right)\cdot\left(X+\mu_{Y(Q)}-\sigma_{Y(Q)}\right)}{\sigma_{Y(Q)}\cdot\mu_{Y(Q)}^2} < 0$$

Since $M(X,Q)$ in (50) is a constant like for the *WHP* contract, the first-order derivative $dQ^{PEN}(X)/dX$ is identical to that in (21).

Buyer decision

The buyer under a PEN contract is compensated for missing units by the penalty rate. The profit the buyer generates is the following

$$\Pi_B^{PEN}(X) = p\cdot E\left[\min\left(D,X,Y(Q)\right)\right] - w\cdot E\left[\min\left(X,Y(Q)\right)\right] + \pi\cdot E\left[\left(X-Y(Q)\right)^+\right]$$

The two cases B(I) ($X \le D$) and B(II) ($X \ge D$) are evaluated in the next section.

Case B(I)

The buyer's profit in case B(I) ($X \le D$) transforms to

$$\Pi_B^{PEN}(X) = (p-w)\cdot E\left[\min\left(X,Y(Q)\right)\right] + \pi\cdot E\left[\left(X-Y(Q)\right)^+\right] = (p-w-\pi)\cdot E\left[\min\left(X,Y(Q)\right)\right] + \pi\cdot X$$

$$\Pi_B^{PEN}(X) = (p-w-\pi)\cdot L(X,Q) + \pi\cdot X \tag{52}$$

with $L(X,Q)$ from (17). Taking the first order derivative yields the expression below[23]

$$\frac{d\Pi_B^{PEN}(X)}{dX} = (p-w-\pi)\cdot\left(1-F_S\left(z_{X,Q}\right)\right) + \pi + (p-w-\pi)\cdot M(X,Q)\cdot\frac{dQ(X)}{dX} \tag{53}$$

[22] For $\partial M(X,Q)/\partial Q$ see Appendix A3.

[23] For more details on the derivative, see Appendix 6.2

with $M(X,Q)$ from (18) and $dQ(X)/dX$ from (21). The optimal order quantity under case B(I), $X_{B(I)}^{PEN}$, then results from $d\Pi_B^{PEN}(X)/dX \overset{!}{=} 0$. Nevertheless, it might be preferable to raise the order quantity above demand ($X \geq D$).

Case B(II)

Under case B(II), the buyer maximizes the subsequent profit

$$\Pi_B^{PEN}(X) = p \cdot E\left[\min(D, Y(Q))\right] - (w+\pi) \cdot E\left[\min(X, Y(Q))\right] + \pi \cdot X$$

$$\Pi_B^{PEN}(X) = p \cdot L(D,Q) - (w+\pi) \cdot L(X,Q) + \pi \cdot X \tag{54}$$

with $L(D,Q)$ from (5) and $L(X,Q)$ from (17). The buyer's decision under case B(II), $X_{B(II)}^{PEN}$, is derived from taking the first order condition $d\Pi_B^{PEN}(X)/dX \overset{!}{=} 0$ from the derivative below [24]

$$\frac{d\Pi_B^{PEN}(X)}{dX} = -(w+\pi) \cdot \left(1 - F_S\left(z_{X,Q}\right)\right) + \pi + \left(p \cdot M(D,Q) - (w+\pi) \cdot M(X,Q)\right) \cdot \frac{dQ(X)}{dX} \tag{55}$$

with $M(D,Q)$ from (7), $M(X,Q)$ from (18) and $dQ(X)/dX$ from (21).

Interaction of buyer and supplier

As under the ORS contract, it has to be analyzed whether there exists a combination of contract parameters which guarantees that total supply chain profit is maximized while both, supplier and buyer, accept the contract. In order to coordinate the supply chain, the optimality conditions of supply chain and supplier under a PEN contract have to be identical. They are given from (8) and (50), respectively:

$\frac{c}{p} = M\left(D, Q^*\right)$ and $\frac{c}{w+\pi} = M\left(X, Q^{PEN}\right)$.

This condition is fulfilled if the buyer orders at demand level, i.e. if $X^{PEN} = D$ and if $M\left(D, Q^*\right) = M\left(X, Q^{PEN}\right)$,i.e. if the following condition for the contract parameters is satisfied

$$p = w + \pi \tag{56}$$

which ensures that $c/p = c/(w+\pi)$. Given the parameter condition, the supplier's marginal profit in (49) turns out to be zero:

$$\frac{d\Pi_S^{PEN}(Q|X)}{dQ} = (w+\pi) \cdot \frac{c}{w+\pi} - c = 0 \text{ .}$$

As the supplier's marginal profit is zero, she actually choses the corresponding input quantity. Because the buyer anticipates this behavior, it can be evaluated which order decision maximizes his profit. Under case B(II) ($X \geq D$), the buyer's marginal profit from (55) in combination with the parameter condition in (56), transforms to

[24] For more details, see Appendix 6.2.

$$\frac{d\Pi_B^{PEN}(X)}{dX} = -(w+\pi)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\pi+\left((w+\pi)\cdot\frac{c}{(w+\pi)}-(w+\pi)\cdot\frac{c}{w+\pi}\right)\cdot\frac{dQ(X)}{dX}$$

$$\frac{d\Pi_B^{PEN}(X)}{dX} = -w+(w+\pi)\cdot F_S\left(z_{X,Q}\right) \tag{57}$$

For proving that $d\Pi_B^{PEN}(X)/dX < 0$, it has to be shown that the penalty π must not be too large. Thus, the determination of the penalty needs particular analysis. Under coordination (given $p = w+\pi$ and $X^{PEN} = D$ which leads to $Q^{PEN} = Q^*$), and using the supply chain profit from (6), the supplier's and the buyer's profits from (48) and (54) can be expressed as follows

$$\Pi_S^{PEN}\left(Q^{PEN}\left|X^{PEN}=D\right.\right)=(w+\pi)\cdot L\left(D,Q^{PEN}\right)-\pi\cdot D-c\cdot Q^{PEN}=p\cdot L\left(D,Q^*\right)-c\cdot Q^*-\pi\cdot D=\Pi_{SC}\left(Q^*\right)-\pi\cdot D$$

and

$$\Pi_B^{PEN}\left(X^{PEN}=D\right)=\pi\cdot D.$$

Consequently, in order for the supplier's participation constraint to hold, i.e. to generate a non-negative profit, the maximum penalty π^+ that results in $\Pi_S^{PEN}\left(Q^{PEN}\left|X^{PEN}=D\right.\right)=0$, is given by

$$\pi^+ = \frac{\Pi_{SC}\left(Q^*\right)}{D} \tag{58}$$

From $\Pi_{SC}^*\left(Q^*\right)=p\cdot\left(1-F_S\left(z_{D,Q}^*\right)\right)\cdot D-\left(p\cdot\theta\cdot F_S\left(z_{D,Q}^*\right)-c\right)\cdot Q^*$ in (10) we get:

$$\pi<\pi^+=p\cdot\left(1-F_S\left(z_{D,Q}^*\right)\right)-\left(p\cdot\theta\cdot F_S\left(z_{D,Q}^*\right)-c\right)\cdot\frac{Q^*}{D}$$

Given the coordinating parameter constellation $p = w+\pi$, the restriction $\pi<\pi^+$ transforms to

$$\pi<(w+\pi)\cdot\left(1-F_S\left(z_{D,Q}^*\right)\right)-\left(p\cdot\theta\cdot F_S\left(z_{D,Q}^*\right)-c\right)\cdot\frac{Q^*}{D}$$

From that we further get

$$(w+\pi)\cdot F_S\left(z_{D,Q}^*\right)-w<-\left(p\cdot\theta\cdot F_S\left(z_{D,Q}^*\right)-c\right)\cdot\frac{Q^*}{D} \tag{59}$$

Under case B(II), from (57), the optimal buyer decision of $X^{PEN} = D$ is only given if

$$\frac{d\Pi_B^{PEN}(X)}{dX} = -w+(w+\pi)\cdot F_S\left(z_{X,Q}\right)<0$$

According to (59) this holds if $p\cdot\theta\cdot F_S\left(z_{D,Q}^*\right)-c>0$.

From (7) and (8) we know that

$$F_S\left(z_{D,Q}^*\right)=\frac{c}{p\cdot\theta}+\frac{\sigma_{Y(Q)}^*}{2\cdot\mu_{Y(Q)}^*}\cdot f_S\left(z_{D,Q}^*\right)$$

so that $p\cdot\theta\cdot F_S\left(z_{D,Q}^*\right)-c=p\cdot\theta\cdot\frac{\sigma_{Y(Q)}^*}{2\cdot\mu_{Y(Q)}^*}\cdot f_S\left(z_{D,Q}^*\right)>0$.

Thus, if the participation constraint for the supplier is fulfilled and if the penalty π is restricted to be lower that π^+, the buyer's optimal order quantity will be $X^{PEN}=D$ in case B(II). Since for $X\leq D$ the first-order derivative in (55) reduces to $d\Pi_B^{PEN}(X)/dX=\pi>0$ the contract coordinating parameter condition $p=w+\pi$ also initiates $X^{PEN}=D$ in case B(I). Thus, analogously to the ORS contract, the PEN contract can enable supply chain coordination because the buyer incentivizes the supplier to produce the supply chain optimal amount by ordering at demand level while the contract parameters are fixed appropriately, i.e. if $p=w+\pi$.

Numerical examples

The data for the numerical study were introduced in the section of the WHP contract. The tables below show the coordinating ability of the PEN contract as well as the impact of changing contract parameters on the split of profits between the buyer and the supplier in the supply chain. As for the two previous contract analyses, the binomial distribution is approximated by a normal distribution which is feasible as orders and production quantities exceed the critical value of 26.67 which guarantees that the condition $Q\cdot\theta\cdot(1-\theta)>5$ holds for success probabilities of $0.25\leq\theta\leq0.75$.[25] The examples below also incorporate the restriction for π from (58).

w	π	$Q^{PEN}=Q^*$	$X^{PEN}=D$	Π_S^{PEN}	Π_B^{PEN}	$\Pi_S^{PEN}+\Pi_B^{PEN}=\Pi_{SC}^*$	Q^{PEN}/D
4,42	9,58	419	100	0	958	958	4,19
5	9	419	100	58	900	958	4,19
6	8	419	100	158	800	958	4,19
7	7	419	100	258	700	958	4,19
8	6	419	100	358	600	958	4,19
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	2	419	100	758	200	958	4,19
13	1	419	100	858	100	958	4,19
14	0	419	100	958	0	958	4,19

Table 7: Effect of changing values for w and π on profit distribution for 25% success probability

[25] Recall chapter 3 of the paper (*Analysis for a centralized supply chain*)

w	π	$Q^{PEN} = Q^*$	$X^{PEN} = D$	Π_S^{PEN}	Π_B^{PEN}	$\Pi_S^{PEN} + \Pi_B^{PEN} = \Pi_{SC}^*$	Q^{PEN}/D
2,23	11,77	215	100	0	1177	1177	2,15
3	11	215	100	77	1100	1177	2,15
4	10	215	100	177	1000	1177	2,15
5	9	215	100	277	900	1177	2,15
6	8	215	100	377	800	1177	2,15
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	2	215	100	977	200	1177	2,15
13	1	215	100	1077	100	1177	2,15
14	0	215	100	1177	0	1177	2,15

Table 8: Effect changing values for w and π on profit distribution for 50% success probability

w	π	$Q^{PEN} = Q^*$	$X^{PEN} = D$	Π_S^{PEN}	Π_B^{PEN}	$\Pi_S^{PEN} + \Pi_B^{PEN} = \Pi_{SC}^*$	Q^{PEN}/D
1,46	12,54	142	100	0	1254	1254	1,42
2	12	142	100	54	1200	1254	1,42
3	11	142	100	154	1100	1254	1,42
4	10	142	100	254	1000	1254	1,42
5	9	142	100	354	900	1254	1,42
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	2	142	100	1054	200	1254	1,42
13	1	142	100	1154	100	1254	1,42
14	0	142	100	1254	0	1254	1,42

Table 9: Effect of changing values for w and π on profit distribution for 75% success probability

5. Conclusion and outlook

The analyses in this paper revealed interesting insights into the area of supply chain coordination through contracts in the case of binomially distributed production yields and deterministic demand. The simple WHP contract fails to coordinate the supply due to double marginalization, while contract types with reward or penalty scheme enable coordinated behavior in the supply chain without violating the actors' participation constraints. However, the ORS contract's ability to coordinate a supply chain depends on the variant that is applied. If a *Pull* type contract (without the delivery of excess units) is used, coordination can be achieved. However, if physical delivery of overstock is allowed (*Push* variant), the contract loses its coordination power. For the PEN contract, however, it can be shown that the design enables SC coordination and, depending on the parameter setting (including a maximum penalty restriction), guarantees an arbitrary profit split. Numerical examples confirmed the analytical findings and where used to illustrate each contract's efficiency to coordinate the supply chain as well as the profit split depending on various parameter combinations.

Compared to the results from Inderfurth and Clemens (2014) for stochastically proportional yields, it is revealed that all contract designs retain their ability or disability to trigger coordination. For the coordinating contract types, *Pull*-ORS and PEN, it furthermore holds that only in cases where the buyer orders exactly at demand level coordination is achieved. Regarding the production decisions, it is found that, as in Inderfurth and Clemens (2014), production input is a multiple of the order size. However, the multiplier is not a constant any longer. Due to the characteristic of binomial yields to decrease in risk as the input size rises, the multiplier changes in every instance of adjusting demand or order sizes (which determine production input decisions). Nevertheless, whether the multiplier increases, decreases or alternates, depends on the critical ratio of contract parameters and demand. In terms of the numerical examples, the supply chain generates a larger maximum profit under binomial (BI) yield than under uniformly distributed stochastically proportional (SP) yield (data as given in section 4.1, mean SP yield rate and BI success probability $\mu_Z = \theta = 0{,}5$: $\Pi_{SC}^{BI} = 1177$ and $\Pi_{SC}^{SP} = 871$, see Inderfurth and Clemens (2014) p. 544). However, the distribution of profits between buyer and supplier is almost the same for all contracts, apart from medium values for the penalty under the PEN contract where the supplier benefits from the higher total supply chain profit under binomial yields.

Further research should focus on extending the supply chain to an emergency option for procuring extra units in case of under-delivery. This option was introduced by Inderfurth and Clemens (2014) and it was shown to coordinate the supply chain by applying the WHP contract (given that only the supplier is utilizing the emergency source). In the current setting, this option may reveal a similar performance. Besides, the setting can also be adjusted with respect to supply chain structure. An important aspect in this context is the extension from a serial to a converging supply chain. Concentrating on further types of yield uncertainty, the all-or-nothing type of yield realization, also known as disruption risk (see Xia et al. (2011)), has hardly received any attention in literature so far.

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Appendix

A1. Details on transformation of delivery/sales quantity *L*

We arrive at the final form of the supply chain's sales quantity $L(D,Q)$ in (5) as shown below.

<u>Definition</u>: $L(D,Q) := E\left[\min\left(D, Y(Q)\right)\right]$

Rearranging yields the following

$$L(D,Q) = D - \int_0^D (D-y)\cdot f_{Y(Q)}(y)dy = D - D\cdot F_{Y(Q)}(D) + \int_0^D y\cdot f_{Y(Q)}(y)dy$$

For normal approximation of binomially distributed yield it holds that $\int_0^D y\cdot f_{Y(Q)}(y)dy = \int_{-\infty}^D y\cdot f_{Y(Q)}(y)dy$.

For normally distributed random variables the following relationship can be utilized to further transform the above expression:[26]

$$\int_{-\infty}^D y\cdot f_{Y(Q)}(y)dy = \mu_{Y(Q)}\cdot F_S\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right) - \sigma_{Y(Q)}\cdot f_S\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right)$$

Replacing the integral and further rearranging yields

$$\begin{aligned} L(D,Q) &= D - \sigma_{Y(Q)}\cdot\left(F_S\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right)\cdot\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right) + f_S\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right)\right) \\ &= D - \sigma_{Y(Q)}\cdot\left(F_S\left(z_{D,Q}\right)\cdot z_{D,Q} + f_S\left(z_{D,Q}\right)\right) \end{aligned}$$

with $F_S(\cdot)$ and $f_S(\cdot)$ as *cdf* and *pdf* of the standard normal distribution and $z_{D,Q} := \left(D-\mu_{Y(Q)}\right)/\sigma_{Y(Q)}$.

The delivery quantity in a decentralized supply chain is identical with the above expression with X instead of D and w instead of p . The term is the following

$$L(X,Q) = E\left[\min\left(X, Y(Q)\right)\right] = X - \sigma_{Y(Q)}\cdot\left(F_S\left(z_{X,Q}\right)\cdot z_{X,Q} + f_S\left(z_{X,Q}\right)\right)$$

A2. First order derivatives of sales/delivery quantity L

In this section the first order derivatives are given for the sales quantity of the supply chain $L(D,Q)$ in (5) and for the delivery quantity of the supplier to the buyer $L(X,Q)$ in (17) with respect to production quantity *Q* as well as order quantity *X*. $\mu_{Y(Q)}$ and $\sigma_{Y(Q)}$ are given from (1) and (2).

<u>Definition</u>: $L(D,Q) := D - \sigma_{Y(Q)}\cdot\left(F_S\left(z_{D,Q}\right)\cdot z_{D,Q} + f_S\left(z_{D,Q}\right)\right) = D - \left(F_S\left(z_{D,Q}\right)\cdot\left(D-\mu_{Y(Q)}\right) + \sigma_{Y(Q)}\cdot f_S\left(z_{D,Q}\right)\right)$

[26] Compare Chopra and Meindl (2012) p. 404.

with $z_{D,Q} := \dfrac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}$

$$\frac{\partial L(D,Q)}{\partial Q} = -\left(\frac{\partial}{\partial Q}\left[F_S\left(z_{D,Q}\right)\cdot\left(D-\mu_{Y(Q)}\right)\right] + \frac{\partial}{\partial Q}\left[\sigma_{Y(Q)}\cdot f_S\left(z_{D,Q}\right)\right]\right)$$

with

$$\frac{\partial z_{D,Q}}{\partial Q} = -\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)}{2\cdot\sigma_{Y(Q)}\cdot\mu_{Y(Q)}}$$

$$\frac{\partial F_S\left(z_{D,Q}\right)}{\partial Q} = \frac{dF_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q} = f_S\left(z_{D,Q}\right)\cdot\left(-\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)}{2\cdot\sigma_{Y(Q)}\cdot\mu_{Y(Q)}}\right)$$

$$\begin{aligned}\frac{\partial f_S\left(z_{D,Q}\right)}{\partial Q} &= \frac{df_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q} = -z_{D,Q}\cdot f_S\left(z_{D,Q}\right)\cdot\left(-\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)}{2\cdot\sigma_{Y(Q)}\cdot\mu_{Y(Q)}}\right)\\ &= \frac{f_S\left(z_{D,Q}\right)}{2}\cdot\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)\cdot\left(D-\mu_{Y(Q)}\right)}{\sigma^2_{Y(Q)}\cdot\mu_{Y(Q)}}\end{aligned}$$

$$\frac{d\mu_{Y(Q)}}{dQ} = \theta$$

Thus, we get

$$\begin{aligned}\frac{d\sigma_{Y(Q)}}{dQ} &= \frac{\theta\cdot(1-\theta)}{2\cdot\sqrt{\theta\cdot(1-\theta)\cdot Q}} = \frac{1}{2}\cdot\sqrt{\frac{\theta\cdot(1-\theta)\cdot Q}{Q^2}} = \frac{\sigma_{Y(Q)}}{2\cdot Q}\\ &= \frac{\sigma_{Y(Q)}\cdot\theta}{2\cdot\mu_{Y(Q)}}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial Q}\left[F_S\left(z_{D,Q}\right)\cdot\left(D-\mu_{Y(Q)}\right)\right] &= \frac{\partial F_S\left(z_{D,Q}\right)}{\partial Q}\cdot\left(D-\mu_{Y(Q)}\right) + F_S\left(z_{D,Q}\right)\cdot\frac{d}{dQ}\left(D-\mu_{Y(Q)}\right)\\ &= -f_S\left(z_{D,Q}\right)\cdot\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)}{2\cdot\sigma_{Y(Q)}\cdot\mu_{Y(Q)}}\cdot\left(D-\mu_{Y(Q)}\right) + F_S\left(z_{D,Q}\right)\cdot(-\theta)\\ &= -\frac{f_S\left(z_{D,Q}\right)\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot\frac{\left(D+\mu_{Y(Q)}\right)\cdot\left(D-\mu_{Y(Q)}\right)}{\sigma_{Y(Q)}} - F_S\left(z_{D,Q}\right)\cdot\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial Q}\left[\sigma_{Y(Q)}\cdot f_S\left(z_{D,Q}\right)\right] &= \frac{d\sigma_{Y(Q)}}{dQ}\cdot f_S\left(z_{D,Q}\right) + \sigma_{Y(Q)}\cdot\frac{df_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q}\\ &= \frac{\sigma_{Y(Q)}\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right) + \sigma_{Y(Q)}\cdot\frac{f_S\left(z_{D,Q}\right)}{2}\cdot\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)\cdot\left(D-\mu_{Y(Q)}\right)}{\sigma^2_{Y(Q)}\cdot\mu_{Y(Q)}}\\ &= \frac{f_S\left(z_{D,Q}\right)\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot\left(\sigma_{Y(Q)} + \frac{\left(D+\mu_{Y(Q)}\right)\cdot\left(D-\mu_{Y(Q)}\right)}{\sigma_{Y(Q)}}\right)\end{aligned}$$

Summarizing, this yields

$$\frac{\partial L(D,Q)}{\partial Q}=-\left(-\frac{f_S\left(z_{D,Q}\right)\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot\frac{\left(D+\mu_{Y(Q)}\right)\cdot\left(D-\mu_{Y(Q)}\right)}{\sigma_{Y(Q)}}-F_S\left(z_{D,Q}\right)\cdot\theta+\frac{f_S\left(z_{D,Q}\right)\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot\left(\sigma_{Y(Q)}+\frac{\left(D+\mu_{Y(Q)}\right)\cdot\left(D-\mu_{Y(Q)}\right)}{\sigma_{Y(Q)}}\right)\right)$$

$$\begin{aligned}\frac{\partial L(D,Q)}{\partial Q}&=-\left(\frac{f_S\left(z_{D,Q}\right)\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot\sigma_{Y(Q)}-F_S\left(z_{D,Q}\right)\cdot\theta\right)\\&=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(z_{D,Q}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right)\right)\end{aligned}$$

For the delivery quantity from the supplier to the buyer, the analysis is identical with X instead of D and w instead of p. Thus, the delivery quantity and its first order derivative are given as follows

$$L(X,Q)=X-\sigma_{Y(Q)}\cdot\left(F_S\left(z_{X,Q}\right)\cdot z_{X,Q}+f_S\left(z_{X,Q}\right)\right)$$

$$\frac{\partial L(X,Q)}{\partial Q}=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(z_{X,Q}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{X,Q}\right)\right)$$

A3. First order derivatives of M

In this section the first order derivatives are given for the right-hand-sides of the supply chain's and the supplier's optimality conditions, respectively, namely $M(D,Q)$ and $M(X,Q)$. $\mu_{Y(Q)}$ and $\sigma_{Y(Q)}$ are given from (1) and (2).

Definition: $M(D,Q):=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(z_{D,Q}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right)\right)$ with $z_{D,Q}:=\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}$

Calculating $\frac{\partial M(D,Q)}{\partial Q}$

$$\frac{\partial M(D,Q)}{\partial Q}=\frac{\theta}{2}\cdot\left(2\cdot\frac{dF_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q}-\frac{d}{dQ}\left[\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right)\right]\right)$$

with $\frac{dF_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q}$, $\frac{df_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q}$, $\frac{d\mu_{Y(Q)}}{dQ}$, and $\frac{d\sigma_{Y(Q)}}{dQ}$ from section A2 in the Appendix.

The second part of the derivative is given by

$$\frac{\partial}{\partial Q}\left[\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right)\right]=\frac{d}{dQ}\left(\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\right)\cdot f_S\left(z_{D,Q}\right)+\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot\frac{df_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial Q}$$

with

$$\frac{d}{dQ}\left(\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\right)=\frac{\frac{d\sigma_{Y(Q)}}{dQ}\cdot\mu_{Y(Q)}-\sigma_{Y(Q)}\cdot\frac{d\mu_{Y(Q)}}{dQ}}{\mu_{Y(Q)}^{2}}$$

$$=\frac{\frac{\sigma_{Y(Q)}\cdot\theta}{2\cdot\mu_{Y(Q)}}\cdot\mu_{Y(Q)}-\sigma_{Y(Q)}\cdot\theta}{\mu_{Y(Q)}^{2}}=-\frac{\sigma_{Y(Q)}\cdot\theta}{2\cdot\mu_{Y(Q)}^{2}}$$

which results in

$$\frac{\partial}{\partial Q}\left[\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right)\right]=-\frac{\sigma_{Y(Q)}\cdot\theta}{2\cdot\mu_{Y(Q)}^{2}}\cdot f_S\left(z_{D,Q}\right)+\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot\frac{f_S\left(z_{D,Q}\right)}{2}\cdot\frac{\theta\cdot\left(D-\mu_{Y(Q)}\right)\cdot\left(D+\mu_{Y(Q)}\right)}{\sigma_{Y(Q)}^{2}\cdot\mu_{Y(Q)}}$$

$$=\frac{f_S\left(z_{D,Q}\right)}{2}\cdot\frac{\theta\cdot\left(D-\mu_{Y(Q)}\right)\cdot\left(D+\mu_{Y(Q)}\right)-\sigma_{Y(Q)}^{2}}{\mu_{Y(Q)}^{2}\cdot\sigma_{Y(Q)}}$$

The total first order derivative $\partial M\left(D,Q\right)/\partial Q$ is given by

$$\frac{\partial M(D,Q)}{\partial Q}=\frac{\theta}{2}\cdot\left(-2\cdot\frac{f_S\left(z_{D,Q}\right)}{2}\cdot\frac{\theta\cdot\left(D+\mu_{Y(Q)}\right)}{\sigma_{Y(Q)}\cdot\mu_{Y(Q)}}-\frac{f_S\left(z_{D,Q}\right)}{2}\cdot\frac{\theta\cdot\left(D-\mu_{Y(Q)}\right)\cdot\left(D+\mu_{Y(Q)}\right)-\sigma_{Y(Q)}^{2}}{\mu_{Y(Q)}^{2}\cdot\sigma_{Y(Q)}}\right)$$

$$=-\frac{\theta}{2}\cdot\frac{f_S\left(z_{D,Q}\right)\cdot\theta}{2}\cdot\frac{2\cdot\mu_{Y(Q)}\cdot\left(D+\mu_{Y(Q)}\right)+\left(D-\mu_{Y(Q)}\right)\cdot\left(D+\mu_{Y(Q)}\right)-\sigma_{Y(Q)}^{2}}{\mu_{Y(Q)}^{2}\cdot\sigma_{Y(Q)}}$$

$$=-f_S\left(z_{D,Q}\right)\cdot\frac{\theta^{2}}{4}\cdot\frac{\left(D+\mu_{Y(Q)}+\sigma_{Y(Q)}\right)\cdot\left(D+\mu_{Y(Q)}-\sigma_{Y(Q)}\right)}{\mu_{Y(Q)}^{2}\cdot\sigma_{Y(Q)}}$$

It is obvious that the above derivative is negative if $\mu_{Y(Q)}>\sigma_{Y(Q)}$.

<u>Proof</u>: For $\theta\cdot Q\geq 1$ (which coincides with the condition for validity of the Normal approximation and holds for all numerical examples throughout the paper)

$$\mu_{Y(Q)}=\theta\cdot Q>\sqrt{\theta\cdot Q}>\sqrt{\theta\cdot(1-\theta)\cdot Q}=\sigma_{Y(Q)}$$

Calculating $\dfrac{\partial M\left(D,Q\right)}{\partial D}$

$$\frac{\partial M\left(D,Q\right)}{\partial D}=\frac{\theta}{2}\cdot\left(2\cdot\frac{dF_S\left(z\right)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial D}-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot\frac{df_S\left(z\right)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial D}\right)$$

with

$$\frac{dF_S\left(z\right)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial D}=f_S\left(z_{D,Q}\right)\cdot\frac{1}{\sigma_{Y(Q)}}$$

and

$$\frac{df_S(z)}{dz}\cdot\frac{\partial z_{D,Q}}{\partial D}=-z_{D,Q}\cdot f_S\left(z_{D,Q}\right)\cdot\frac{1}{\sigma_{Y(Q)}}=-f_S\left(z_{D,Q}\right)\cdot\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}^2}$$

The total first order derivative $\partial M(D,Q)/\partial D$ is given by

$$\begin{aligned}\frac{\partial M(D,Q)}{\partial D}&=\frac{\theta}{2}\cdot\left(2\cdot\frac{f_S\left(z_{D,Q}\right)}{\sigma_{Y(Q)}}-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot\left(-f_S\left(z_{D,Q}\right)\cdot\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}^2}\right)\right)=\frac{\theta}{2}\cdot\frac{f_S\left(z_{D,Q}\right)}{\sigma_{Y(Q)}}\cdot\left(2+\frac{D-\mu_{Y(Q)}}{\mu_{Y(Q)}}\right)\\&=\frac{\theta}{2}\cdot\frac{f_S\left(z_{D,Q}\right)}{\sigma_{Y(Q)}}\cdot\left(1+\frac{D}{\mu_{Y(Q)}}\right)\end{aligned}$$

As all terms in the above derivative are positive, it follows that $\partial M(D,Q)/\partial D>0$.

The above analysis can again be used for calculating the first order derivatives of the supplier's right hand side of the production decision's optimality conditions with X instead of D and w instead of p. The results are shown below:

$$M(X,Q):=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(z_{X,Q}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{X,Q}\right)\right)\text{ with } z_{X,Q}:=\frac{X-\mu_{Y(Q)}}{\sigma_{Y(Q)}}$$

$$\frac{\partial M(X,Q)}{\partial Q}=-f_S\left(z_{X,Q}\right)\cdot\frac{\theta^2}{4}\cdot\frac{\left(X+\mu_{Y(Q)}+\sigma_{Y(Q)}\right)\cdot\left(X+\mu_{Y(Q)}-\sigma_{Y(Q)}\right)}{\mu_{Y(Q)}^2\cdot\sigma_{Y(Q)}}<0$$

$$\frac{\partial M(X,Q)}{\partial X}=\frac{\theta}{2}\cdot\frac{f_S\left(z_{X,Q}\right)}{\sigma_{Y(Q)}}\cdot\left(1+\frac{X}{\mu_{Y(Q)}}\right)>0$$

A4. Proof of Lemma

Proof that $\lim_{D\to\infty}\frac{Q}{D}=\frac{1}{\theta}$ with Q from optimality condition in (8): $\frac{c}{p}=M(D,Q)$

Recall $M(D,Q):=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(z_{D,Q}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(z_{D,Q}\right)\right)$; $z_{D,Q}:=\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}$ and $\mu_{Y(Q)}$ and $\sigma_{Y(Q)}$ from (1) and (2).

$$\begin{aligned}\frac{c}{p}&=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right)-\frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}}\cdot f_S\left(\frac{D-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right)\right)\\&=\frac{\theta}{2}\cdot\left(2\cdot F_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)-\frac{\sqrt{\theta\cdot(1-\theta)\cdot Q}}{\theta\cdot Q}\cdot f_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)\right)\end{aligned}$$

Proving the validity of the above condition for $D\to\infty$ can be done in two steps:

Step 1) Proving that $\lim_{D\to\infty}\frac{\sqrt{\theta\cdot(1-\theta)\cdot Q}}{\theta\cdot Q}\cdot f_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)=0$:

$$\frac{\sqrt{\theta\cdot(1-\theta)\cdot Q}}{\theta\cdot Q}\cdot f_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)=\sqrt{\frac{1-\theta}{\theta}}\cdot\frac{1}{\sqrt{Q}}\cdot f_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)$$

From (12) we know that $dQ(D)/dD>0$ and hence, $\lim_{D\to\infty}Q=\infty$. Consequently, $\lim_{D\to\infty}\frac{1}{\sqrt{Q}}=0$. If that holds, the expression under Step 1) approaches zero since $f_S\left(z_{D,Q}\right)$ is bounded from above: $\lim_{D\to\infty}\sqrt{\frac{1-\theta}{\theta}}\cdot\frac{1}{\sqrt{Q}}\cdot f_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)=0$.

Step 2) Proving that $\frac{c}{p}=\theta\cdot F_S\left(\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)$ only holds if $\lim_{D\to\infty}\frac{Q}{D}=\frac{1}{\theta}$ holds:

Considering $\frac{D-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}$ from the above expression and dividing by *D* yields

$$\frac{(D-\theta\cdot Q)\cdot 1/D}{\sqrt{\theta\cdot(1-\theta)}\cdot\frac{Q}{\sqrt{Q}}\cdot 1/D}=\frac{1-\theta\cdot Q/D}{Q/D\cdot\sqrt{\theta\cdot(1-\theta)}\cdot\frac{1}{\sqrt{Q}}}.$$

Thus,

$$\frac{c}{p}=\theta\cdot F_S\left(\frac{1-\theta\cdot Q/D}{Q/D\cdot\sqrt{\theta\cdot(1-\theta)}\cdot\frac{1}{\sqrt{Q}}}\right)$$

Further transforming yields

$$\frac{1-\theta\cdot Q/D}{Q/D\cdot\sqrt{\theta\cdot(1-\theta)}\cdot\frac{1}{\sqrt{Q}}}=F_S^{-1}\left(\frac{c}{p\cdot\theta}\right)$$

$$\frac{1-\theta\cdot Q/D}{Q/D}=F_S^{-1}\left(\frac{c}{p\cdot\theta}\right)\cdot\sqrt{\theta\cdot(1-\theta)}\cdot\frac{1}{\sqrt{Q}}$$

From Step 1) above we know that $\lim_{D\to\infty}Q=\infty$ and $\lim_{D\to\infty}\frac{1}{\sqrt{Q}}=0$. Consequently, the right hand side of the above equation is zero if *D* approaches infinity. In order for the left hand side to approach zero, the following condition has to hold:

$$1-\theta\cdot Q/D=0$$

$$\frac{Q}{D}=\frac{1}{\theta}$$

Hence, it is proven that if D approaches infinity, the production quantity Q approaches $D\cdot 1/\theta$.

A5. Characteristics of distribution and density function of standard normal distribution

Considering the cumulated distribution function (*cdf*) of the standard normal distribution, for $Q=X$ it follows

$$F_S\left(z_{X,Q}\right)=F_S\left(\frac{X-\mu_{Y(Q)}}{\sigma_{Y(Q)}}\right)=F_S\left(\frac{Q-\theta\cdot Q}{\sqrt{\theta\cdot(1-\theta)\cdot Q}}\right)=F_S\left(\sqrt{\frac{1-\theta}{\theta}\cdot Q}\right).$$

The *cdf* of the standard normal distribution ranges from $-\infty$ to $+\infty$. However, intersecting the ordinate at 0.5 (i.e $F_S(0)=0.5$), it approaches 1 quickly. For values as low as 4, it is approximately 1 ($F_S(4)=0.99997$[27]). Considering our numerical examples with success probabilities in $0.25\leq\theta\leq 0.75$, the assumption holds for production quantities of $Q\geq 48$ which is given for all our examples.

The probability density function (*pdf*) of the standard normal distribution can be analyzed analogously. For $Q=X$, it holds that

$$f_S\left(z_{X,Q}\right)=f_S\left(\sqrt{\frac{1-\theta}{\theta}\cdot Q}\right)$$

Again, it can be shown that for large values, the *pdf* takes an extreme value, in this case zero. Given $f_S(4)=0.00013$, production quantities larger than 48 allow us to assume that the *pdf* approaches zero if $Q=X$ holds.

A6. Detailed analyses of first order derivatives of selected profit functions under ORS and PEN contract

6.1 ORS contract

The general form for calculating the first order derivative is given by

$$\frac{d\Pi_B^{ORS}(X)}{dX}=\frac{\partial\Pi_B^{ORS}(X)}{\partial X}+\frac{\partial\Pi_B^{ORS}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX}$$ with $dQ(X)/dX$ from (21).

Case B(I)

From (38) the buyer's profit is given by

$$\Pi_B^{ORS}(X)=(p-w+w_O)\cdot L(X,Q)-w_O\cdot\mu_{Y(Q)}$$

Calculation of first order derivative:

$$\frac{\partial\Pi_B^{ORS}(X)}{\partial X}=(p-w+w_O)\cdot\frac{\partial L(X,Q)}{\partial X}=(p-w+w_O)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)$$

and

[27] Consult tables of standard normal distribution or Excel worksheet function.

$$\frac{\partial \Pi_B^{ORS}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX}=\left((p-w+w_O)\cdot\frac{\partial L(X,Q)}{\partial Q}-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$
$$=\left((p-w+w_O)\cdot M(X,Q)-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$

with $\partial L(X,Q)/\partial X$ from (25) and $\partial L(X,Q)/\partial Q$ from (18). Finally, the total first order derivative in (39) is given by

$$\frac{d\Pi_B^{ORS}(X)}{dX}=(p-w+w_O)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\left((p-w+w_O)\cdot M(X,Q)-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$

Case B(II)

From (40) the buyer's profit is given by

$$\Pi_B^{ORS}(X)=p\cdot L(D,Q)-(w-w_O)\cdot L(X,Q)-w_O\cdot\mu_{Y(Q)}$$

Calculation of first order derivative:

$$\frac{\partial \Pi_B^{ORS}(X)}{\partial X}=-(w-w_O)\cdot\frac{\partial L(X,Q)}{\partial X}=-(w-w_O)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)$$
$$\frac{\partial \Pi_B^{ORS}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX}=\left(p\cdot\frac{\partial L(D,Q)}{\partial Q}-(w-w_O)\cdot\frac{\partial L(X,Q)}{\partial Q}-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$
$$=\left(p\cdot M(D,Q)-(w-w_O)\cdot M(X,Q)-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$

with $\partial L(X,Q)/\partial X$ from (25), $\partial L(D,Q)/\partial Q$ from (7) and $\partial L(X,Q)/\partial Q$ from (18). Finally, the total first order derivative in (41) is given by

$$\frac{d\Pi_B^{ORS}(X)}{dX}=-(w-w_O)\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\left(p\cdot M(D,Q)-(w-w_O)\cdot M(X,Q)-w_O\cdot\theta\right)\cdot\frac{dQ(X)}{dX}$$

Interaction of buyer and supplier

$$\Pi_{SC}^*(X)=p\cdot L\left(X,Q^*\right)-c\cdot Q^*$$

$$\frac{\Pi_{SC}^*(X)}{dX}=\frac{\partial\Pi_{SC}^*(X)}{\partial X}+\frac{\partial\Pi_{SC}^*(X)}{\partial Q}\cdot\frac{dQ(X)}{dX}$$
$$\frac{\partial\Pi_{SC}^*(X)}{\partial X}=p\cdot\frac{\partial L(X,Q)}{\partial X}=p\cdot\left(1-F_S\left(z_{X,Q}\right)\right)$$
$$\frac{\partial\Pi_{SC}^*(X)}{\partial Q}\cdot\frac{dQ(X)}{dX}=\left(p\cdot\frac{\partial L(X,Q)}{\partial Q}-c\right)\cdot\frac{dQ(X)}{dX}=\left(p\cdot M(X,Q)-c\right)\cdot\frac{dQ(X)}{dX}$$

$$\frac{d\Pi_{SC}^*(X)}{dX}=p\cdot\left(1-F_S\left(z_{X,Q}\right)\right)+\left(p\cdot M(X,Q)-c\right)\cdot\frac{dQ(X)}{dX}$$

with $\partial L(X,Q)/\partial X$ from (25) and $\partial L(X,Q)/\partial Q$ from (18).

6.2 PEN contract

The general form for calculating the first order derivative is given by

$$\frac{d\Pi_B^{PEN}(X)}{dX} = \frac{\partial\Pi_B^{PEN}(X)}{\partial X} + \frac{\partial\Pi_B^{PEN}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX}$$ with $dQ(X)/dX$ from (21).

Case B(I)

From (52) the buyer's profit is given by

$$\Pi_B^{PEN}(X) = (p-w-\pi)\cdot L(X,Q) + \pi\cdot X$$

Calculation of first order derivative:

$$\frac{\partial\Pi_B^{PEN}(X)}{\partial X} = (p-w-\pi)\cdot\frac{\partial L(X,Q)}{\partial X} + \pi = (p-w-\pi)\cdot\left(1-F_S\left(z_{X,Q}\right)\right) + \pi$$

$$\begin{aligned}\frac{\partial\Pi_B^{PEN}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX} &= (p-w-\pi)\cdot\frac{\partial L(X,Q)}{\partial Q}\cdot\frac{dQ(X)}{dX}\\ &= (p-w-\pi)\cdot M(X,Q)\cdot\frac{dQ(X)}{dX}\end{aligned}$$

with $\partial L(X,Q)/\partial X$ from (25) and $\partial L(X,Q)/\partial Q$ from (18). Finally, the total first order derivative in (53) is given by

$$\frac{d\Pi_B^{PEN}(X)}{dX} = (p-w-\pi)\cdot\left(1-F_S\left(z_{X,Q}\right)\right) + \pi + (p-w-\pi)\cdot M(X,Q)\cdot\frac{dQ(X)}{dX}$$

Case B(II)

From (54) the buyer's profit is given by

$$\Pi_B^{PEN}(X) = p\cdot L(D,Q) - (w+\pi)\cdot L(X,Q) + \pi\cdot X$$

Calculation of first order derivative:

$$\frac{\partial\Pi_B^{PEN}(X)}{\partial X} = -(w+\pi)\cdot\frac{\partial L(X,Q)}{\partial X} + \pi = -(w+\pi)\cdot\left(1-F_S\left(z_{X,Q}\right)\right) + \pi$$

and

$$\begin{aligned}\frac{\partial\Pi_B^{PEN}(X)}{\partial Q}\cdot\frac{dQ(X)}{dX} &= \left(p\cdot\frac{\partial L(D,Q)}{\partial Q} - (w+\pi)\cdot\frac{\partial L(X,Q)}{\partial Q}\right)\cdot\frac{dQ(X)}{dX}\\ &= \left(p\cdot M(D,Q) - (w+\pi)\cdot M(X,Q)\right)\cdot\frac{dQ(X)}{dX}\end{aligned}$$

with $\partial L(X,Q)/\partial X$ from (25), $\partial L(D,Q)/\partial Q$ from (7) and $\partial L(X,Q)/\partial Q$ from (18). Hence, we get the total first order derivative in (55) as

$$\frac{d\Pi_B^{PEN}(X)}{dX} = -(w+\pi)\cdot\left(1-F_S\left(z_{X,Q}\right)\right) + \pi + \left(p\cdot M(D,Q) - (w+\pi)\cdot M(X,Q)\right)\cdot\frac{dQ(X)}{dX}$$

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