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Defuzzification in Scenario Management – A theoretical and practical Guide

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Defuzzification in Scenario Management –

A theoretical and practical Guide

Thomas Spengler and Sebastian Herzog[1]

Abstract

It is virtually a truism that companies should respond to situations of high complexity, dynamics and contingency with strategic management measures. Not least in turbulent VUCA-type environments, the company is well advised to use alternative scenarios for environmental forecasting. The so-called scenario technique is used for this purpose. Human modes of thought, judgment and action are often ambiguous and rarely univocal. Here, so-called expert (control) systems based on fuzzy control offer valuable help. Their core is rule inference resulting in fuzzy rule output. In many cases, however, managers are interested in receiving concrete recommendations for action that are as unambiguous as possible. To do so, they need to defuzzify the fuzzy output set. In this paper, we consider selected defuzzification procedures in the area of scenario management.

JEL: A20, A22, A23, C60, M20, M21

Keywords: scenario management, defuzzification, fuzzy logic

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Contents

1 Introduction

1.1 Preliminary remarks

A frequently and rightly recommended instrument of strategic management, which strictly speaking is not a singular tool but a toolbox, is the so-called scenario technique (Chermack/Lynham/Ruona 2001, Clemons 1995, Dess/Lumpkin/Eisner 2007, Godet 1987, 1995, 2001, Godet/Roubelat 1996, Grant 2008, Heijden 1996, Hill/Schilling/Jones 2016, Kluyver/Pearce II 2003, Lindgren/Bandhold 2003, Linnemann/Klein 1985, Reibnitz 1995, Roubelat 2000, Schoemaker/Heijden 1992, Wack 1985a, 1985b, Wilson 1998), for which various action models have been developed. Among these action models are rule-based expert systems (Volkmer/Metzger/Spengler/Vogt 2019), where a rule is understood as an if-then linkage. The if-component is also referred to as rule premise or rule input, the then-component as rule conclusion or rule output. The if-then link is established by a so-called inference (Comesana Benavides/Prado 2002, Zimmermann 1996). Scenarios are future development paths of data constellations relevant for decision-making (Georgantzas/ Acar 1995, Heijden 1997, 2000, Huss/Honton 1987, Kleiner 1999, Ringland 1998, Schoemaker 1993, 1995, Schwartz 1991, Simpson 1992). Since the number of potential scenarios can quickly grow immeasurably in real cases, their number must be reduced to a few. In the literature, it is often recommended to ultimately use those scenarios for the development of strategic alternatives that are as consistent, relevant and probable as possible (Amara/Lipinski 1983, Chandler/Cockler 1982, Chermack/Lynham/Ruona 2001, Heijden 1997).

However, there are problems associated with the probabilistic assessment of scenarios (Brauers/Weber 1988, Fahey/Randall 1998, Kluyver/Moskowitz 1984):

The total probability mass is always 1 and is to be distributed among the selected scenarios. With four scenarios and equal distribution, the scenario probability is 0.25 (this is certainly not high). If the probability of a scenario is estimated with 0.8, a mass of 0.2 remains for the remaining scenarios. We see therefore that the mentioned requirement is not to be fulfilled. We therefore argue here against probabilistic approaches and recommend the use of (modal-logically founded) possibilistic models based on fuzzy set and possibility or necessity theory (Canarelli 1996, Dubois 1988, Dubois/Lang/Prade 1987, 1991, Dubois/Prade 1987, 1988, 2001, Zadeh 1978). At their core, they offer two advantages:

On the one hand, one does not get into the probabilistic conflict described above, and on the other hand, the use of possibilistic calculations corresponds particularly well to human modes of thinking. Thus, for example, one does not have to deal with the question of how likely one

assesses innovative product developments through the use of digital technologies in the next twenty years, but how possible one considers them to be then, or, to use another example, how one assesses the price development of fossil and renewable energy sources possibilistically for the medium-term future. While one can hardly answer the first question seriously, one is much more capable of estimating the corresponding possibility values.

The common methods of the classical scenario technique are based on the Boolean logic, according to which an element x uniquely belongs or uniquely does not belong to a set A.

Thus, for the membership value of such a crisp set A, $\mu_A(x) \in \{0,1\}$ holds.

In the context of fuzzy logic (Buckley/Eslami 2002, Gottwald 1993, Pedrycz 1993, Piegat 2001, Zadeh 1983, Zimmermann 1987, 1996), on the other hand, the membership of an element x in a fuzzy set $\tilde{A}$ can also take on graded values (between 0 and 1), such that $\mu_{\tilde{A}}(x) \in [0,1]$ holds (Bellmann/Zadeh 1970, Dubois/Ostasiewicz/Prade 2000, Dubois/Prade 1980a, Pedrycz 1993, Piegat 2001, Wang/Chang 2000, Zimmermann 1996).

The approaches and tools of fuzzy logic have been developed not least since the seminal work of Zadeh (1965) and are now widely used, especially in the engineering field. Although many applications - think, for example, of fuzzy logic-based crane and elevator controls, camcorders or washing machines - are now part of the standard repertoire of control engineering, the management sciences have to date still had some difficulty with this set of tools (Darwin/Johnson/McAuley 2002). This is especially regrettable insofar as fuzzy calculations are particularly appropriate for human thinking, because which manager, for example, will want to, let alone be able to formulate exact scenarios (in the sense of 0-1 logic) for a ten-, fifteen- or even twenty-year planning horizon.

1.2 Influence analysis, consistency analysis and cross-impact analysis

Key scenario management tools include (a) influence analysis, (b) consistency analysis, and (c) cross-impact analysis.

Ad (a): In the context of influence analysis, which we do not want to discuss further here, one determines spheres of influence that are disaggregated into influence factors and for which one determines so-called influence scores (Reibnitz 1995).

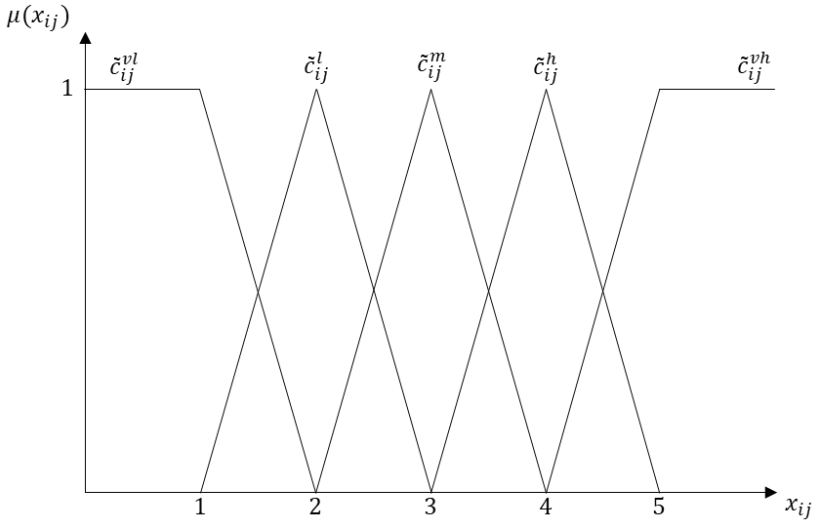
The corresponding influence scores (b_{ij}) express the surveyed experts' assessment of the extent to which factor i exerts an influence on factor j. For example, the potential influence can be measured on a 6-point scale from $0 =$ no influence, $1 =$ very low influence, $2 =$ low influence,

$3 =$ medium influence, $4 =$ high influence to $5 =$ very high influence. Depending on the underlying set of values, e.g. $b_{ij} \in \{0,1,2,3,4,5\}$ or $b_{ij} \in [0,1]$ may then apply. By appropriate addition over j (resp. i) one then arrives at corresponding active (resp. passive) sums, which in turn then allow a rational reduction of the descriptors. While in the crisp case, one calculates with crisp values (b_{ij}), in the fuzzy case one uses fuzzy numbers or intervals, which can be processed just as easily, but determined more realistically.

Ad (b): In the case of the consistency analysis (Kluyver/Pearce II 2003), critical descriptors and their expressions must first be examined in pairs with regard to their consistency. If, for example, three descriptors each have two expressions, six pairs of descriptors are to be assessed. These are then amalgamated to so-called assumption bundles and also checked for consistency. Again, crisp and fuzzy calculi can be used, whereas in the fuzzy case fuzzy numbers (Dubois/Prade 1978, Spengler/Vogt 2008), fuzzy intervals or so-called linguistic variables (Zadeh 1975, 1987) can be considered. The latter represent quadruples that include the set of linguistic terms in addition to the name of the linguistic variable, the corresponding basic set, and a semantic rule (Zadeh 1975). For example, a linguistic variable represents the sales potential of a company. The corresponding basic set then consists of potential sales figures, which can be measured in pieces, monetary units, or weights, among others. The semantic rule assigns a membership function to each linguistic term on the corresponding base set, while the linguistic terms designate possible expressions of the linguistic variables (e.g. "low", "medium", "high"). In the course of the consistency analysis, each pair of descriptors i and j is to be assessed in terms of its consistency. For example, members of a scenario team measure consistency on a continuous (and here: crisp[2]) input set $x_{ij} \in [0,6]$.Moreover, the extent of the respective consistency ($\tilde{c}$) could be assessed via the linguistic terms "very low" (vl), "low" (l), "medium" (m), "high" (h), and "very high" (vh). Each of these terms is then assigned a membership function. Fuzzy numbers and fuzzy intervals are particularly suitable for this purpose. While the membership function of a fuzzy number has a clear peak at the 1-level and to the left or right of it a rising or falling function course, the 1-level of the fuzzy interval represents a plateau. In the further course, we want to assume linear membership functions throughout here, since these are particularly easy to process and correspond to the human modes of thinking in many cases.

[2] The use of fuzzy input values is possible without major problems. However, we do not want to consider them further here for the purpose of complexity reduction.

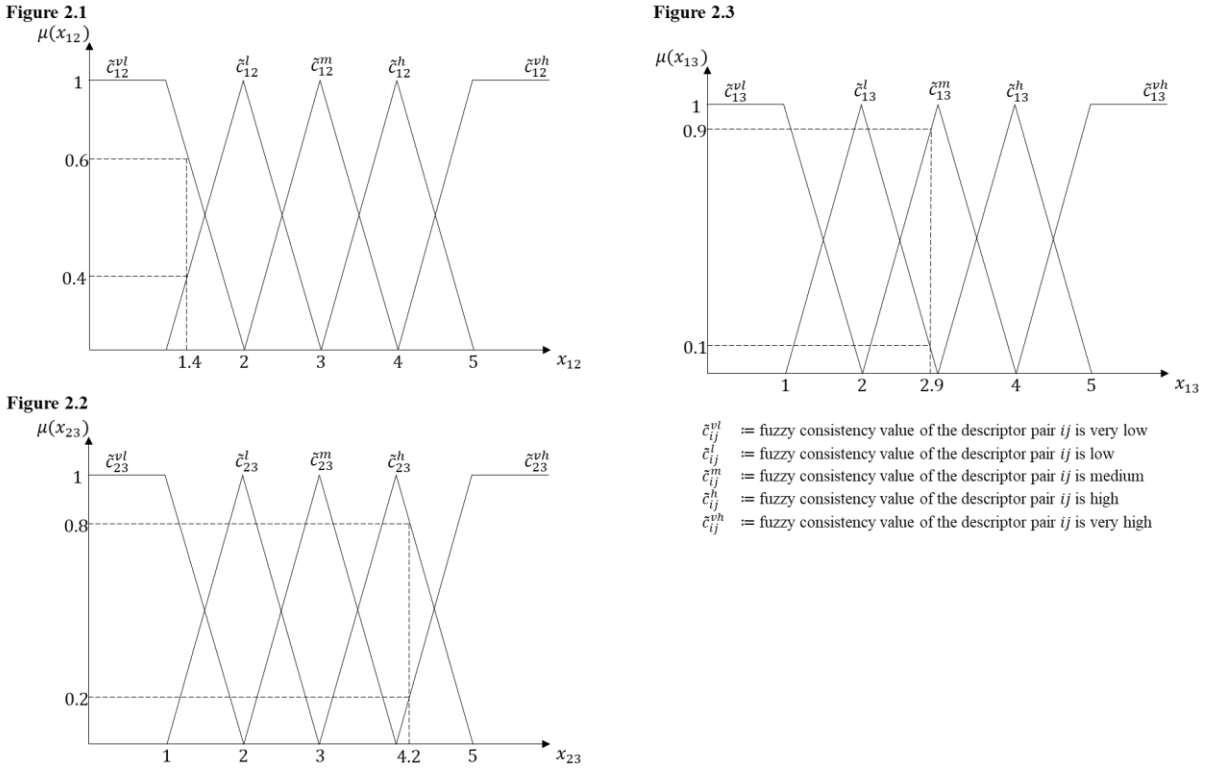
Figure 1 shows, among other things, that in the case of continuous linear membership functions, the experts provide the following judgments: They consider input values x_{ij} between 0 and 1 as "very low" in any case (membership value $\mu(x_{ij}) = 1$) and those equal to 2 as not "very low" at all (membership value $\mu(x_{ij}) = 0$). They do not consider input values between 1 and 2 to be completely "very low" ($1 > \mu(x_{ij}) > 0$). They judge an input value equal to $x_{ij} = 2$ as "low" in any case, input values equal to 1 and 3 as "low" in no case, and input values between 1 and 2 and between 2 and 3 as graded "low". The membership functions of further input values are to be interpreted analogously.



$\tilde{c}_{ij}^{vl}$:= fuzzy consistency value of the descriptor pair ij is very low
$\tilde{c}_{ij}^{l}$:= fuzzy consistency value of the descriptor pair ij is low
$\tilde{c}_{ij}^{m}$:= fuzzy consistency value of the descriptor pair ij is medium
$\tilde{c}_{ij}^{h}$:= fuzzy consistency value of the descriptor pair ij is high
$\tilde{c}_{ij}^{vh}$:= fuzzy consistency value of the descriptor pair ij is very high

Figure 1: Graphical representation of the fuzzy expert judgments

It can be seen from figures 2.1-2.3 that our experts estimate the consistency of the descriptor pair $i = 1$ and $j = 2$ with x_{12} and thus judge it to the degree 0.6 as "very low" and to the degree 0.4 as "low". In addition, x_{13} as well as x_{23} apply, so that the consistency of the descriptor pair 1 and 3 to degree 0.9 is rated as "medium" as well as to degree 0.1 as "low" and that of the descriptor pair 2 and 3 to degree 0.8 as "high" as well as to degree 0.2 as "very high".



Figures 2.1-2.3: Descriptor expressions with membership values to fuzzy consistency values

The inference rule of modus (ponendo) ponens known from Boolean propositional logic (Dubois/Prade 1991, Zimmermann 1987) is also frequently applied in fuzzy logic-based expert systems (Hall/Kandel 1991, Zimmermann 1996). For the form of fuzzy consistency analysis of interest here, this means that (if-then) rules for the conclusions to be drawn from the premises (here: fuzzy consistency values of descriptor pairs) are to be derived for the overall consistency of entire assumption bundles. In the above example, (for example) the rules can be formulated as follows:

Rule 1:

$x_{12} = \tilde{c}_{12}^{vl} \wedge x_{13} = \tilde{c}_{13}^{l} \wedge x_{23} = \tilde{c}_{23}^{l} \rightarrow \tilde{C}_{123} = \text{low}$

Rule 2:

$x_{12} = \tilde{c}_{12}^{l} \wedge x_{13} = \tilde{c}_{13}^{m} \wedge x_{23} = \tilde{c}_{23}^{h} \rightarrow \tilde{C}_{123} = \text{medium}$

Here $\tilde{C}_{123}$ symbolizes the overall consistency of the descriptor triple 1, 2, and 3. For the subsequent fuzzy inference (Bouchon-Meunier 1991, Dubois/Prade 1991, Piegat 2001, Schneider/Kandel 1991, Zadeh 1983) only those rules (Yager 1991) are further used whose degree of fulfillment (DOF) is positive. This can only be positive if the left side of the implication is positive and is often determined via the so-called minimum operator. In the above example, rule 1 has a DOF of 0 and rule 2 has $DOF = \min(0.4; 0.9; 0.8) = 0.4$, i.e. rule 2 is satisfied to the degree 0.4.

Triangular norms (t-norms) and t-conorms (s-norms) can be used for linking rule components (e.g. input components). t-norms serve to form the set average and thus to link via the logical and ("both … and"), while s-norms serve to unify sets and thus to link via the logical or ("either ... or … or both") (Dubois/Prade 1980b, Fodor/Yager 2000, Pap 2002, Yager 1980). Let T be a certain operator from the class of t-norms (e.g. the minimum operator) then the following properties hold (Klement/Mesiar/Pap 2004, Zimmermann 1996):

$T[x, y]$: $[0,1\,] \times [0,1\,] \rightarrow [0,1\,]$ (Definition range)
$T(x, y) = T(y, x)$ (Commutativity)
$T(T(x, y), z) = T(x, T(y, z)) =$ (Associativity)
$x \leq y \Rightarrow T(x, z) \leq T(y, z)$ (Monotony)
$T(x, 1) = x,\ T(x, 0) = 0$ (Neutral and zero element)

The t-norms include, for example, the minimum operator, the algebraic product, the bounded difference, the drastic product, and the Yager average (Zimmermann 1996). The s-norms, which include the maximum operator, the algebraic sum, the bounded sum, the drastic sum, and the Yager union (Zimmermann 1996), are also commutative, associative, and monotonic. However, for the neutral and the zero element applies $S(x, 1) = 1,\ S(x, 0) = x$, with S as operator from the class of s-norms (Klement/Mesiar/Pap 2004, Zimmermann 1996).

As a simple example, use the following rule to illustrate the background:

"IF it was dry before AND it rains OR snows afterwards, THEN the weather changes". For the AND-link of the if-component, one use a t-norm and for the OR-link, one use a s-norm.

Ad (c): In the course of cross impact analyses (Gordon/Hayward 1968, Sarin 1978) cross influences between the so-called critical descriptors are determined.[3]

For example, the time lag between the occurrence of two descriptors is assessed, which descriptor follows another in time, whether one descriptor is (to which extent) causal for the occurrence of another, etc. In some approaches, (isolated) probabilities of occurrence for individual descriptors and (joint) probabilities for complete scenarios are also determined. However, with the above arguments, we advocate the use of occurrence possibilities instead of occurrence probabilities.

[3] Descriptors whose development cannot be predicted with certainty are referred to as critical in scenario management.

The construction and use of fuzzy logic based expert systems is also suitable for this purpose. Here, analogous to the procedure outlined above, the entry sensitivities of individual descriptor pairs are first assessed and then those of entire scenarios. The overall consistency and the overall possibility of each scenario can then be used to infer their overall relevance, respectively, in order to finally determine the set of scenarios to be selected. Given three descriptors i, j, and k and the linguistic terms vl, l, m, h, and vh for overall consistency ($\tilde{C}_{ijk}$) and l, m, and h for overall possibility ($\tilde{P}_{ijk}$), and vl, l, m and h for the degrees of relevance of the scenarios ($\tilde{R}_{ijk}$) can be determined based on the following rules (see table 1):

Rule	$\tilde{C}_{ijk}$	$\tilde{P}_{ijk}$	$\tilde{R}_{ijk}$
1	vl	l	vl
2	vl	m	vl
3	vl	h	vl
4	l	l	l
5	l	m	l
6	l	h	l
7	m	l	l
8	m	m	m
9	m	h	m
10	h	l	m
11	h	m	m
12	h	h	h
13	vh	l	m
14	vh	m	h
15	vh	h	h

Table 1: Rule system for determining relevance levels of scenarios

It is often recommended to defuzzify the fuzzy conclusions of the rule base (Piegat 2001) in order to arrive at unambiguous recommendations for action. The following example from the private everyday life of an apartment tenant shall serve as an explanation:

He has an (implicit) fuzzy control system for regulating his heating. For example, one of the rules recommends him to set the heating thermostat to "quite high". In order to find out which temperature should be set concretely, defuzzification of the fuzzy rule conclusion is useful. How to perform such defuzzification will be discussed in the next section.

2 Defuzzification methods

2.1 Determination of the fuzzy output set

Let $X = \{x\}$ be a crisp base set of input values, $x_1 \in X$ and $x_2 \in X$ two crisp input values, $\widetilde{A_1}$, $\widetilde{A_2}$ and $\widetilde{A_3}$ three linguistic input variables, and $\mu_{\tilde{A}_1}(x)$, $\mu_{\tilde{A}_2}(x)$ and $\mu_{\tilde{A}_3}(x)$ the corresponding membership functions. Besides, $x_1 = \tilde{A}_2$ and $x_2 = \tilde{A}_3$ are applied (see figure 3):

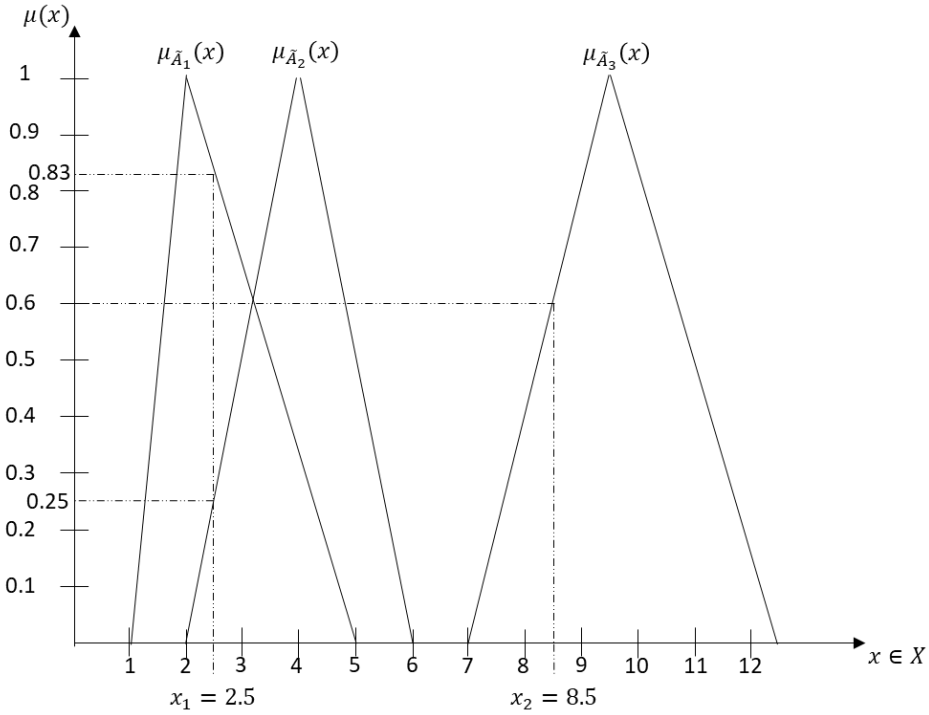


Figure 3: Sample membership functions with selected membership values

For the DOF, if the minimum operator is chosen, applies:

$$DOF = Min(\mu_{\tilde{A}_2}(x_1), \mu_{\tilde{A}_3}(x_2)) = Min(0.25; 0.6) = 0.25$$

For each active rule, the membership function $\mu_{C'}(y)$ of the respective inference is then created. For these applies again, when the minimum operator is used:

$\mu_{C'}(y) = T\big(DOF; \mu_{C'}(y)\big) = \min(DOF; \mu_{C'}(y))$

For example, an active rule could be: IF $x_1 = \tilde{A}_2$ THEN $y = \tilde{C}_2$

The following then holds for the train membership values $\mu_{C'}(y)$, given the input membership function from figure 3:

$\mu_{C'}(y) = Min\big(0.25; \mu_{C'}(y)\big)\ \forall y \in Y$ (see figure 4):

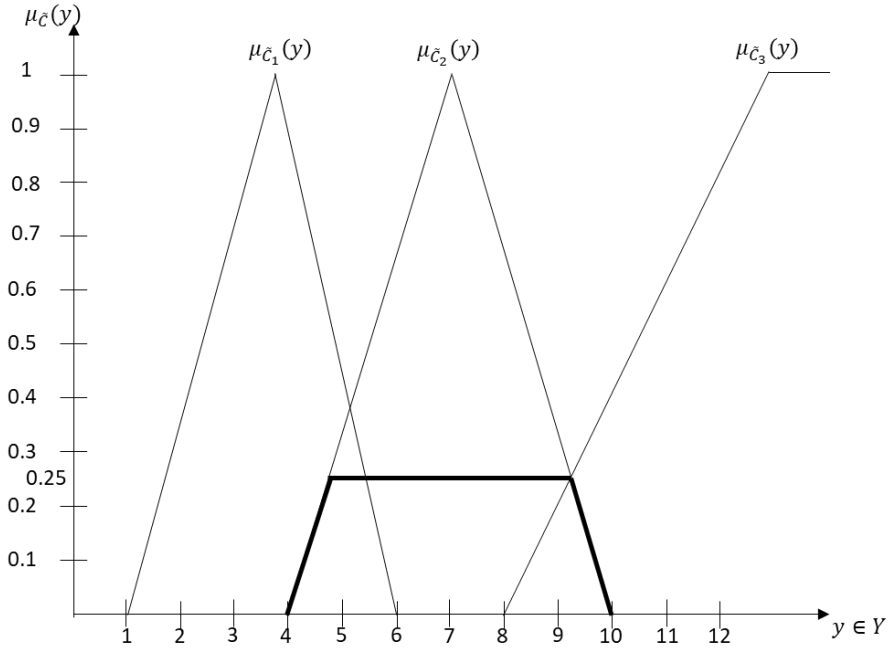


Figure 4: Membership functions of fuzzy outputs

It can be seen that the membership function of the resulting output quantity of a single rule is cut off at the level of $DOF = 0.25$. If several rules are active ($DOF > 0$), the resulting output set and its membership function must be formed over all active rules. The operators commonly used here include those from the area of s-norms. If, in addition to the already mentioned rule „IF $x_1 = \tilde{A}_2$ THEN $y = \tilde{C}_2$“ the rule „IF $x_1 = \tilde{A}_1$ THEN $y = \tilde{C}_1$“ is active, then, taking into account figure 3:

$$\mu_{conc}(y) = S\left(\mu_{\tilde{C}_1,}(y); \mu_{\tilde{C}_2,}(y)\right)$$

and when using the maximum operator:

$$\mu_{conc}(y) = Max\left(\mu_{\tilde{C}_1,}(y); \mu_{\tilde{C}_2,}(y)\right)$$

An example is given in section 3. Once the fuzzy output set has been determined, it can be defuzzified (Van Leekwijck/Kerre 1999). Among others, maximum methods (Pedrycz 1993) (2.2) and the center of gravity method (Pedrycz 1993, Piegat 2001) (2.3) can be considered.

2.2 Maximum methods

Maximum methods include (a) the First-of-Maxima method, (b) the Last-of-Maxima method, (c) the Random-Choice-of-Maxima method, and (d) the Maximum-Mean method.

Ad (a): In this method, the lowest (or worst) abscissa value is chosen from the set of maximum abscissa values (see x^* in figure 5) (Piegat 2001).

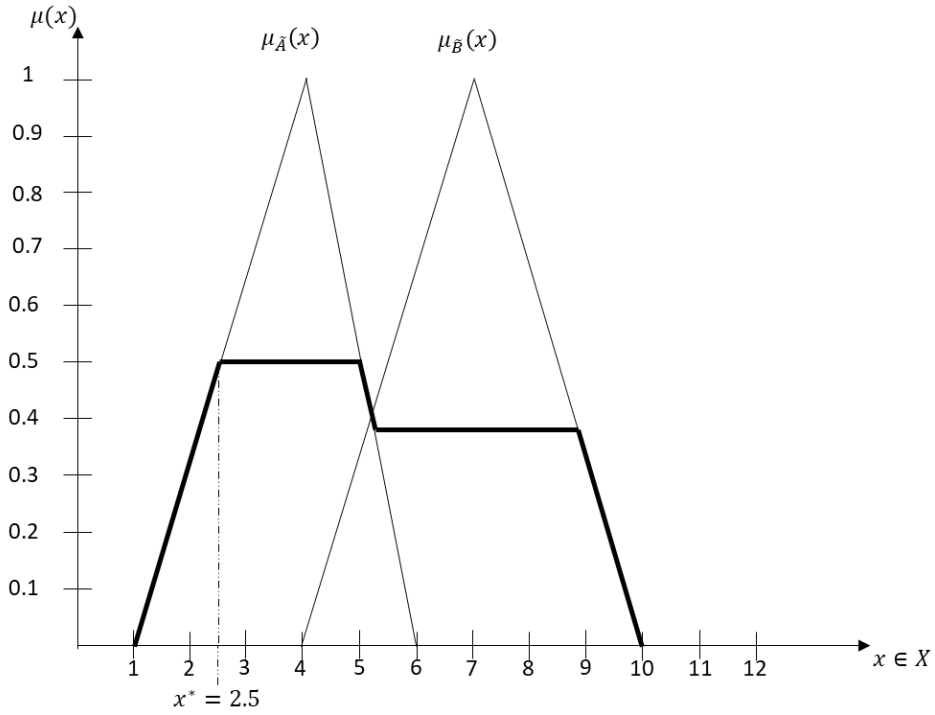


Figure 5: Application of the First-of-Maxima method

Ad (b): Here, the highest (or best) abscissa value is chosen from the set of maximum abscissa values (see x^* in figure 6).

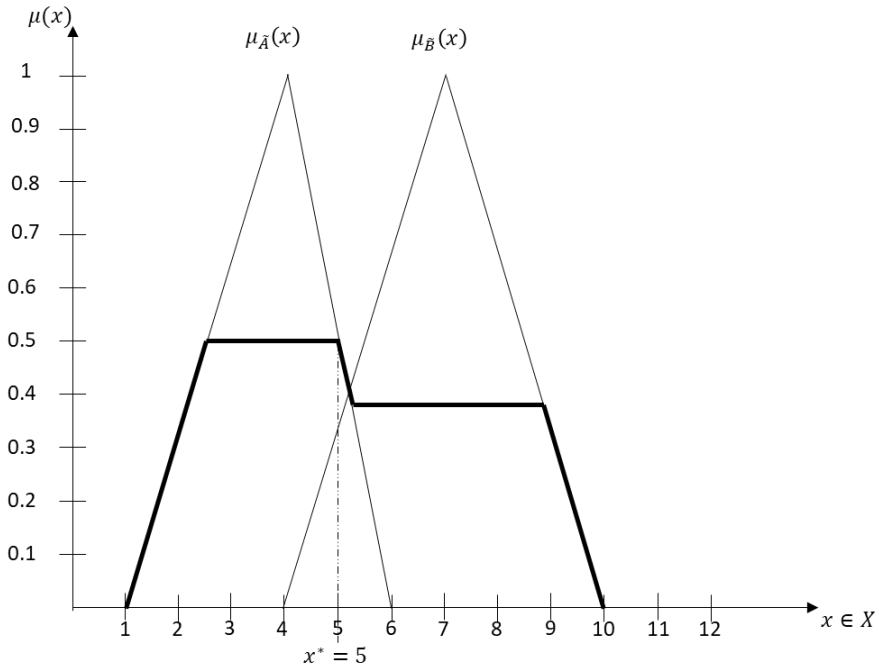


Figure 6: Application of the Last-of-Maxima method

Ad (c): Here, a suitable random mechanism is used to choose an abscissa value from the set of maximum abscissa values and thus any abscissa value between x_u^* and x_o^* (see figure 7).

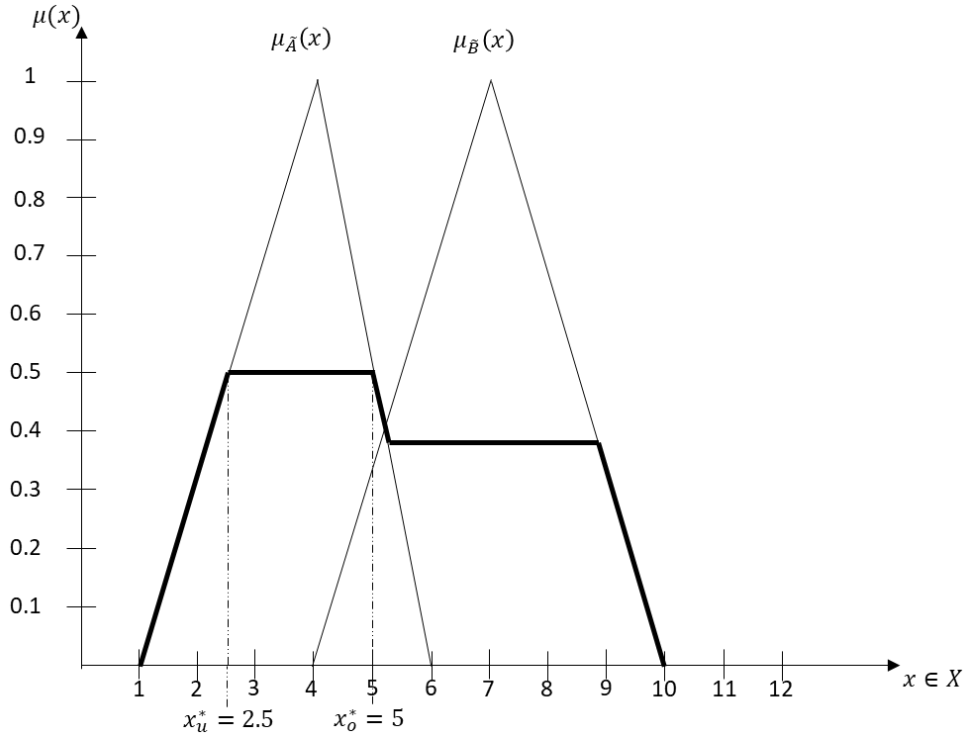


Figure 7: Representation of the Random-Choice-of-Maxima method

Ad (d): In this procedure, one chooses the arithmetic mean of the maximum abscissa values to defuzzify the output set (see x^* in figure 8, with $x^* = \frac{x_u^* + x_o^*}{2}$) (Piegat 2001).

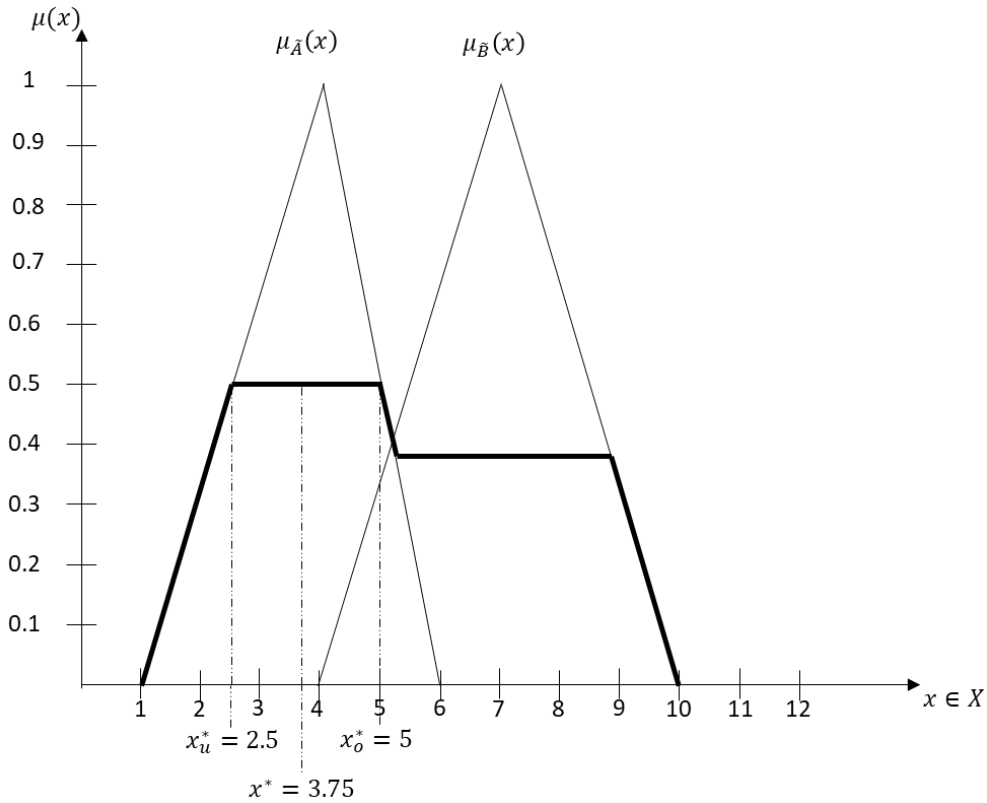


Figure 8: Representation of the Maximum-Mean-Method

The maximum methods offer the advantage of relatively simple defuzzification. Whether to use them or to resort to more complex methods has to be decided depending on the situation.

2.3 Center-of-Gravity method

The center of gravity of a area can be understood as its "center point". Let us imagine a seesaw on a children's playground, which has two arms of equal length. If two children of the same weight (let the mass of a child be m_j with $j = 1$ and $j = 2$ and $m_1 = m_2$; to simplify matters, the children are not considered as bodies but as points) are teetering on the seesaw, the seesaw will come into balance exactly when it is positioned exactly in the center (see figure 9).

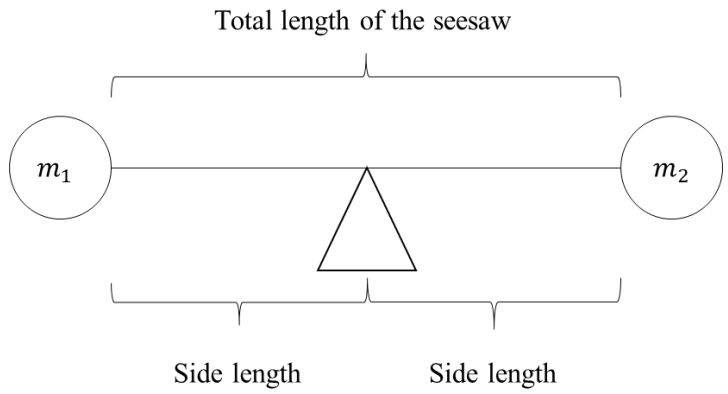


Figure 9: Graphic representation of a seesaw with center of gravity in the middle

However, if one of the children is heavier than the other (e.g., $m_1 > m_2$), the seesaw center of gravity moves toward the heavier child (see figure 10).

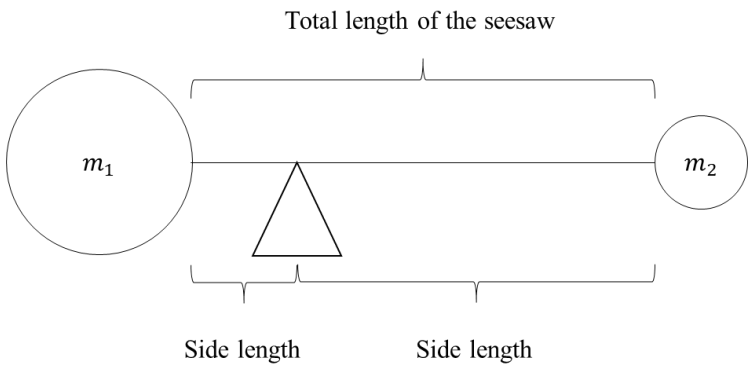


Figure 10: Graphical representation of a seesaw with shifted center of gravity

In the physical sense, the center of gravity of a geometric figure can be interpreted as its center of mass. The center of gravity of the graph of a membership function is then the center of mass of the membership values. We are now looking for the centroid of area of the resulting membership function of the control output for the purpose of defuzzification. In order to be able to compute centroids, one must determine first of all the contents of the area. As is well known,

integral calculus is used for this purpose, especially for (at least partially) curved function graphs.

To determine the area of a area A (e.g. large rectangles in figure 11), it is divided into infinitesimal strips dA and the integral is formed over these strips.

According to this:

$$A = \int dA \tag{1}$$

To do this, A is divided into individual strips, one parallel to the abscissa and the other parallel to the ordinate. For the strips parallel to the abscissa, the area dA of the strips is obtained from the product of their height x and their width dy, while for the strips parallel to the ordinate, the corresponding area dA is obtained from the product of their height y and their width dx (see figure 11).

For the following statements applies in our notation $y = f(x) = \mu(x)$.

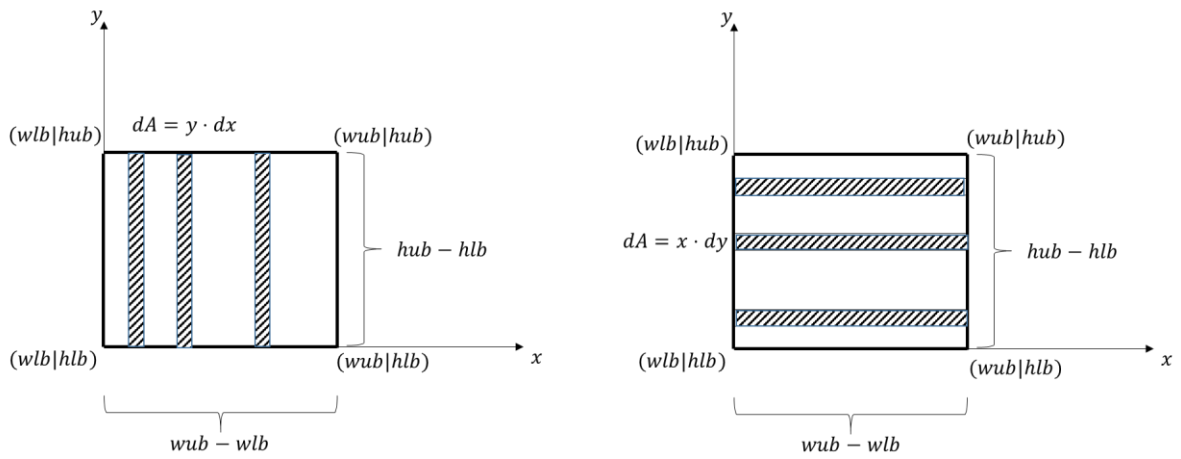


Figure 11: Graphical representation of basic considerations for integral calculus

The membership function of a fuzzy interval is trapezoidal (see figure 12).

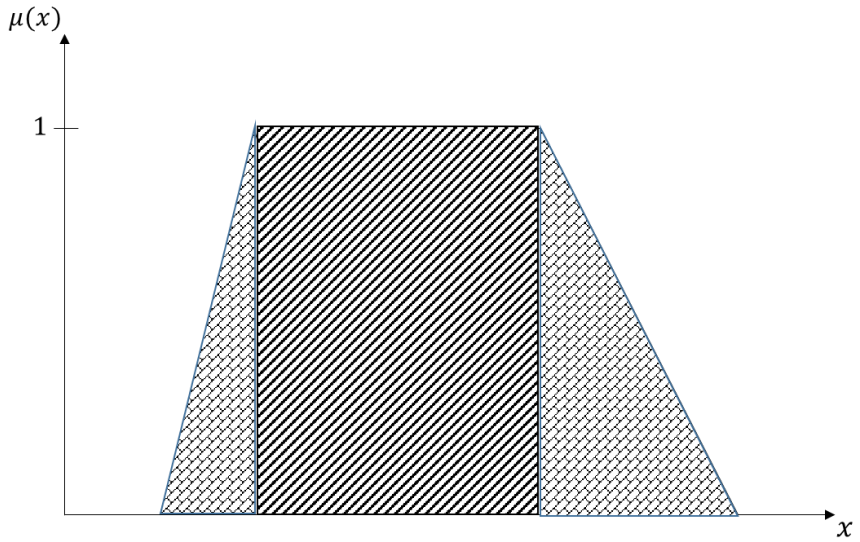


Figure 12: Graphical representation of an area-decomposed fuzzy interval

Such a trapezoid can be decomposed into a rectangle and two triangles. When using fuzzy numbers, the graph of the membership function is exclusively triangular. If the triangular area is also decomposed into strips, rectangles are obtained which end either above or below the membership function. In the first case, the rectangle includes an excess part of the relevant area, whereas in the second case, a relevant piece of area is missing (see figure 13).

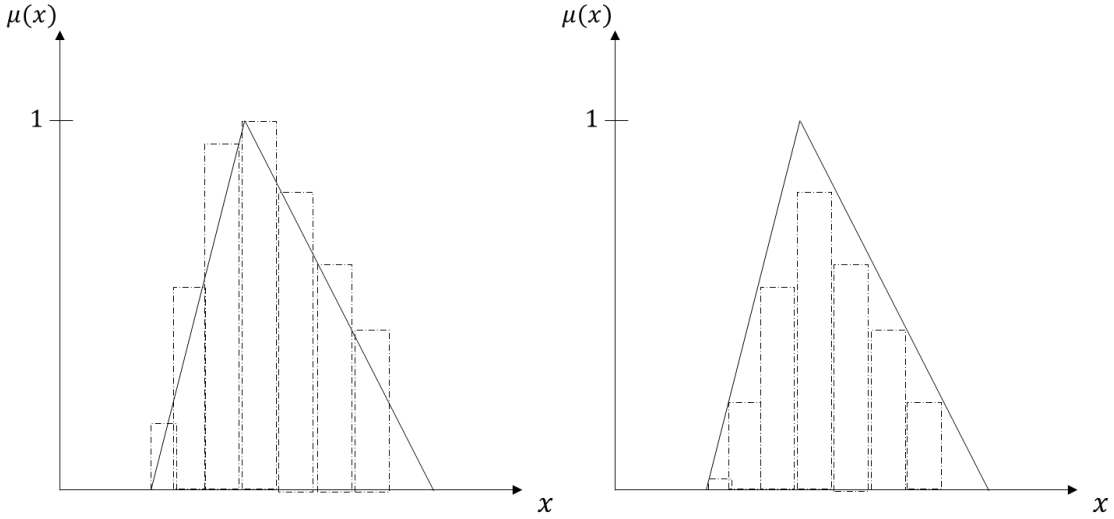


Figure 13: Decomposition of a fuzzy number into pieces

The width of the horizontal and vertical stripes is then let become infinitesimal, so that one arrives at dx and dy, respectively.

Thus, to determine an integral with integration over the abscissa, $dA = y\ dx$ and over the ordinate $dA = x\ dy$ holds.

The following equations can be used to determine the coordinates of the center of area $(x_{cog}|y_{cog})$ and with recourse to the boundaries of the exemplary rectangles ($wub - wlb$ respectively $hub - hlb$) shown above:

$$x_{cog} = \frac{\int x dA}{\int dA} = \frac{\int_{wlb}^{wub} x dA}{\int_{wlb}^{wub} dA} \tag{2}$$

$$y_{cog} = \frac{\int y dA}{\int dA} = \frac{\int_{hlb}^{hub} y dA}{\int_{hlb}^{hub} dA} \tag{3}$$

Here $\int dA$ corresponds to the area of all infinitesimal strips and $\int x\ dA$ respectively $\int y\ dA$ to the area of all infinitesimal strips to be formed, taking into account the distance of the centroid x respectively y of the subarea from the reference edge.

The centroid of the area thus results from the quotient of the sum of the partial areas weighted with x respectively y on the one hand and the total area on the other.

Referring back to figure 11, it is valid for the computational equations to determine the abscissa coordinate $\int dA = \int y\,dx$ and $\int x\,dA = \int x\,y\,dx$, respectively the ordinate coordinate $\int dA = \int x\,dy$ and $\int y\,dA = \int y\,x\,dy$.

Above calculation equations can be adapted accordingly to

$$x_{cog} = \frac{\int x dA}{\int dA} = \frac{\int x\,y\,dx}{\int y\,dx} \tag{4}$$

for the determination of the abscissa coordinate respectively to

$$y_{cog} = \frac{\int y dA}{\int dA} = \frac{\int y\,x\,dy}{\int x\,dy} \tag{5}$$

for the determination of the ordinate coordinate.

Table 2 summarizes the basic computational equations for determining centroid of area coordinates.

Determination of the abscissa coordinate of the centroid of the area	Determination of the ordinate coordinate of the centroid of the area
The area results from	
$A = \int dA$ and with	
$dA = y\,dx$	$dA = x\,dy$
applies:	
$\int dA = \int y\,dx$	$\int dA = \int x\,dy$
$\int x\,dA = \int x\,y\,dx$	$\int y\,dA = \int y\,x\,dy$
and thus for the determination of the coordinates:	
$x_{cog} = \frac{\int x\,dA}{\int dA} = \frac{\int x\,y\,dx}{\int y\,dx}$	$y_{cog} = \frac{\int y\,dA}{\int dA} = \frac{\int y\,x\,dy}{\int x\,dy}$

Table 2: Summary of the basic calculation equations for the determination of centroidal coordinates of areas

If the area under consideration is a rectangle with the shape shown in figure 11, the following simplification can be used, which results from the fact that the width of the rectangle ($wub - wlb$) and the height of the rectangle ($hub - hlb$) are constant and vary neither in the ordinate nor in the abscissa direction. Accordingly, $x_{cog} = \frac{\int x\,y\,dx}{\int y\,dx}$ can be substituted with $x_{cog} = \frac{\int x\,(hub-hlb)dx}{\int (hub-hlb)\,dx}$ respectively $y_{cog} = \frac{\int y\,x\,dy}{\int x\,dy}$ with $y_{cog} = \frac{\int y\,(wub-wlb)\,dy}{\int (wub-wlb)\,dy}$.

Given an assumed height ($hub - hlb$) and an assumed width ($wub - wlb$) of a rectangle, we then integrate over the integral boundaries wlb and wub respectively hlb and hub in the abscissa direction respectively in the ordinate direction.

$$x_{cog} = \frac{\int x dA}{\int dA} = \frac{\int x\, y\, dx}{\int y\, dx} = \frac{\int x\,(hub-hlb)dx}{\int (hub-hlb)dx} = \frac{\frac{1}{2}x^2(hub-hlb)|_{wlb}^{wub}}{x(hub-hlb)|_{wlb}^{wub}} =$$

$$= \frac{\frac{1}{2}wub^2(hub-hlb)-\frac{1}{2}wlb^2(hub-hlb)}{wub(hub-hlb)-wlb(hub-hlb)} \tag{6}$$

$$y_{cog} = \frac{\int y dA}{\int dA} = \frac{\int y\, x\, dy}{\int x\, dy} = \frac{\int y\,(wub-wlb)dy}{\int (wub-wlb)dy} = \frac{\frac{1}{2}y^2(wub-wlb)|_{hlb}^{hub}}{y(wub-wlb)|_{hlb}^{hub}} =$$

$$= \frac{\frac{1}{2}hub^2(wub-wlb)-\frac{1}{2}hlb^2(wub-wlb)}{hub(wub-wlb)-hlb(wub-wlb)} \tag{7}$$

3 Exemplary determination of the centroid of area of a fuzzy output set for fuzzy input sets with continuous triangular or trapezoidal membership functions

3.1 Preliminary remarks

We will now assume here two fuzzy input sets in terms of fuzzy numbers $\tilde{A}$ and $\tilde{B}$, whose (triangular) membership functions are as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-\frac{5-x}{3} & f\ddot{u}r\ 2 \le x \le 5 \\ 1-\frac{x-5}{4} & f\ddot{u}r\ 5 < x \le 9 \\ 0 & otherwise \end{cases} \tag{8.1}$$

$$\mu_{\tilde{B}}(x) = \begin{cases} 1-\frac{11-x}{3} & f\ddot{u}r\ 8 \le x \le 11 \\ 1-\frac{x-11}{2} & f\ddot{u}r\ 11 < x \le 13 \\ 0 & otherwise \end{cases} \tag{9.1}$$

Their transformation leads to:

$$\mu_{\tilde{A}}(x) = \begin{cases} -\frac{2}{3}+\frac{1}{3}x & f\ddot{u}r\ 2 \le x \le 5 \\ \frac{9}{4}-\frac{1}{4}x & f\ddot{u}r\ 5 < x \le 9 \\ 0 & otherwise \end{cases} \tag{8.2}$$

$$\mu_{\tilde{B}}(x) = \begin{cases} -\frac{8}{3}+\frac{1}{3}x & f\ddot{u}r\ 8 \le x \le 11 \\ \frac{13}{2}-\frac{1}{2}x & f\ddot{u}r\ 11 < x \le 13 \\ 0 & otherwise \end{cases} \tag{9.2}$$

The graph of the membership function of the resulting fuzzy output set $\mu_{conc}(x)$ follows the bold line in figure 14 in our example.

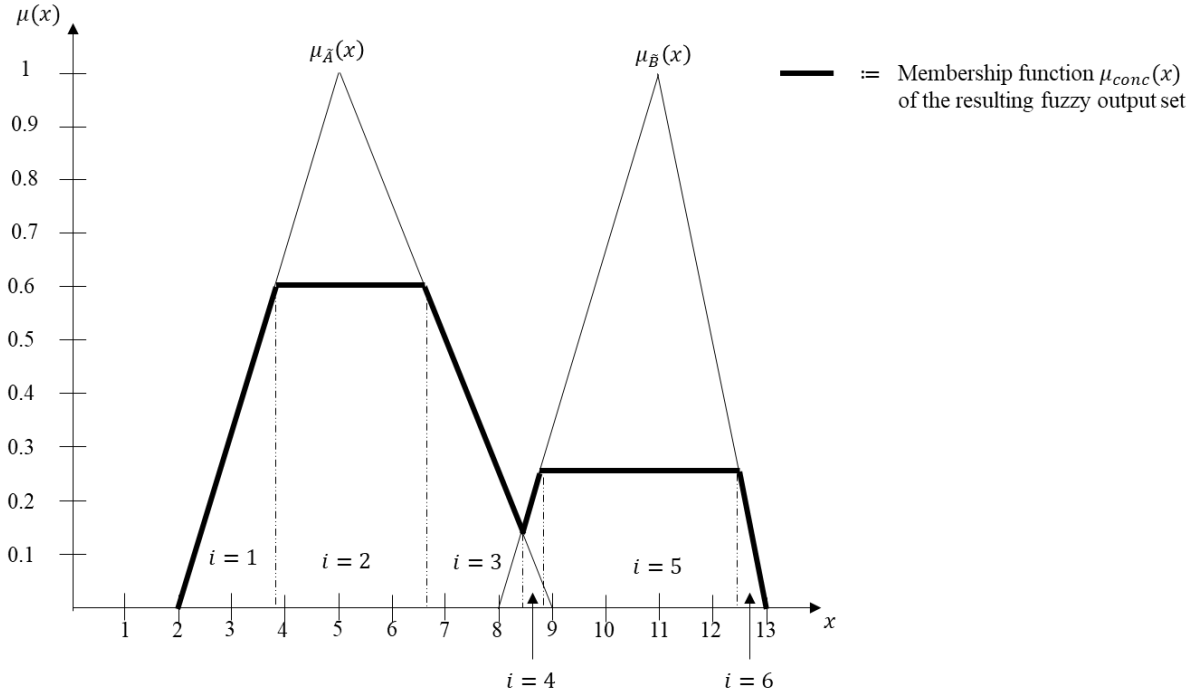


Figure 14: Graphical representation of the membership functions

As shown in figure 14, the entire membership function can be divided into six areas i with $\bar{I} := \{i | i = 1, \dots, 6\}$ (triangles and rectangles).

3.2 Determination of centroid of area on the basis of integral calculus

3.2.1 Determination of the abscissa coordinate of the centroid of the area

The general equation of determination for calculating the abscissa coordinate of a centroid of a single area is:

$$x_{cog} = \frac{\int x dA}{\int dA} = \frac{\int x\, y\, dx}{\int y\, dx} \tag{4}$$

Recalling the use of a notation in terms of membership functions, this leads to the following determination equation:

$$x_{cog} = \frac{\int_{wlb}^{wub} x \cdot \mu_{conc}(x)\, dx}{\int_{wlb}^{wub} \mu_{conc}(x) dx} \tag{10.1}$$

Since the presented fuzzy output set is characterized by six different function courses, a decomposition of the integral into six partial integrals is performed. With recourse to the use of a notation in the sense of membership functions, this leads to the following equation of determination of individual centroids:

$$x_{cog}^{i} = \frac{\int_{wlb}^{wub} x^{i} \cdot \mu_{conc}(x)\, dx}{\int_{wlb}^{wub} \mu_{conc}(x) dx} \quad \forall i \in \bar{I} \tag{10.2}$$

Following the determination of individual area centroids, the identified expressions for numerator and denominator of the quotient shown above can be aggregated to determine x_{cog}:

$$x_{cog} = \frac{\int_{wlb}^{wub} x^i \cdot \mu_{conc}(x)\, dx}{\int_{wlb}^{wub} \mu_{conc}(x) dx} = \frac{\sum_{i=1}^{6} \left[\int_{wlb}^{wub} x \cdot \mu_{conc}(x) dx\right]_i}{\sum_{i=1}^{6} \left[\int_{wlb}^{wub} \mu_{conc}(x) dx\right]_i} \tag{10.3}$$

For each partial integral, we first determine the abscissa coordinate of the centroid of the area:

Section $i = 1$:

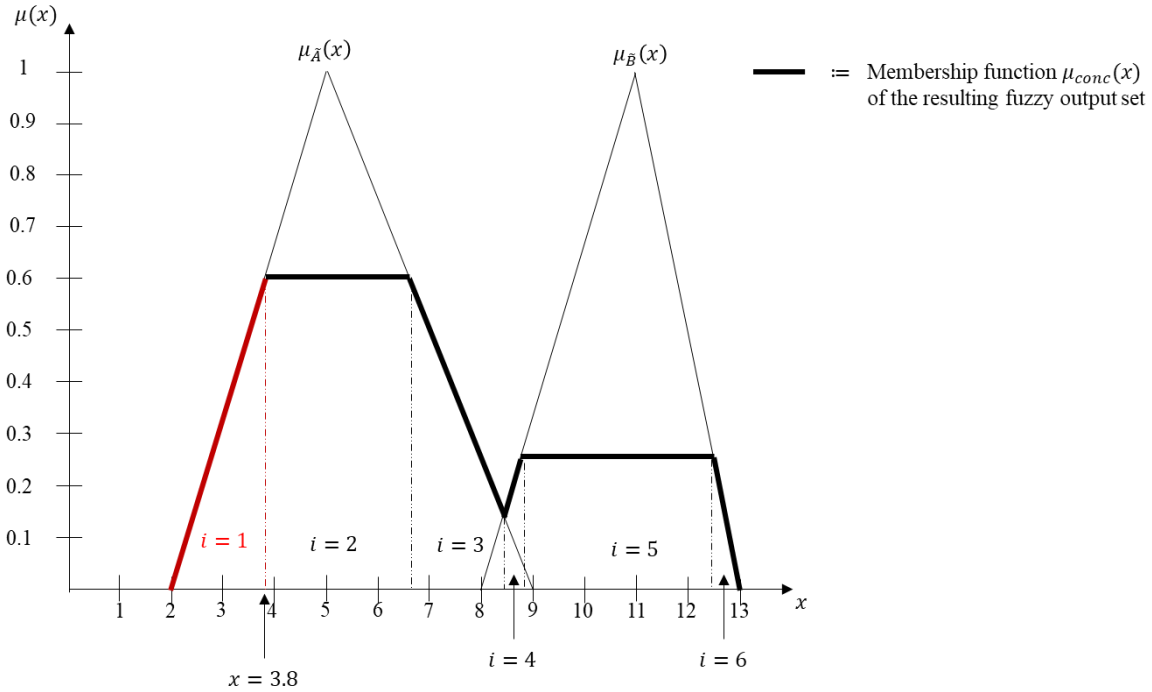


Figure 15: Consideration of section $i = 1$

$$x_{cog}^{i=1} = \frac{\int_2^{3.8} x \cdot \left(-\frac{2}{3} + \frac{1}{3}x\right) dx}{\int_2^{3.8} \left(-\frac{2}{3} + \frac{1}{3}x\right) dx} = \frac{-\frac{1}{3}x^2 + \frac{1}{9}x^3 |_2^{3.8}}{-\frac{2}{3}x + \frac{1}{6}x^2 |_2^{3.8}}$$

$$= \frac{\left(-\frac{1}{3} \cdot 3.8^2 + \frac{1}{9} \cdot 3.8^3\right) - \left(-\frac{1}{3} \cdot 2^2 + \frac{1}{9} \cdot 2^3\right)}{\left(-\frac{2}{3} \cdot 3.8 + \frac{1}{6} \cdot 3.8^2\right) - \left(-\frac{2}{3} \cdot 2 + \frac{1}{6} \cdot 2^2\right)} = \frac{1.728}{0.54}$$

Section $i = 2$:

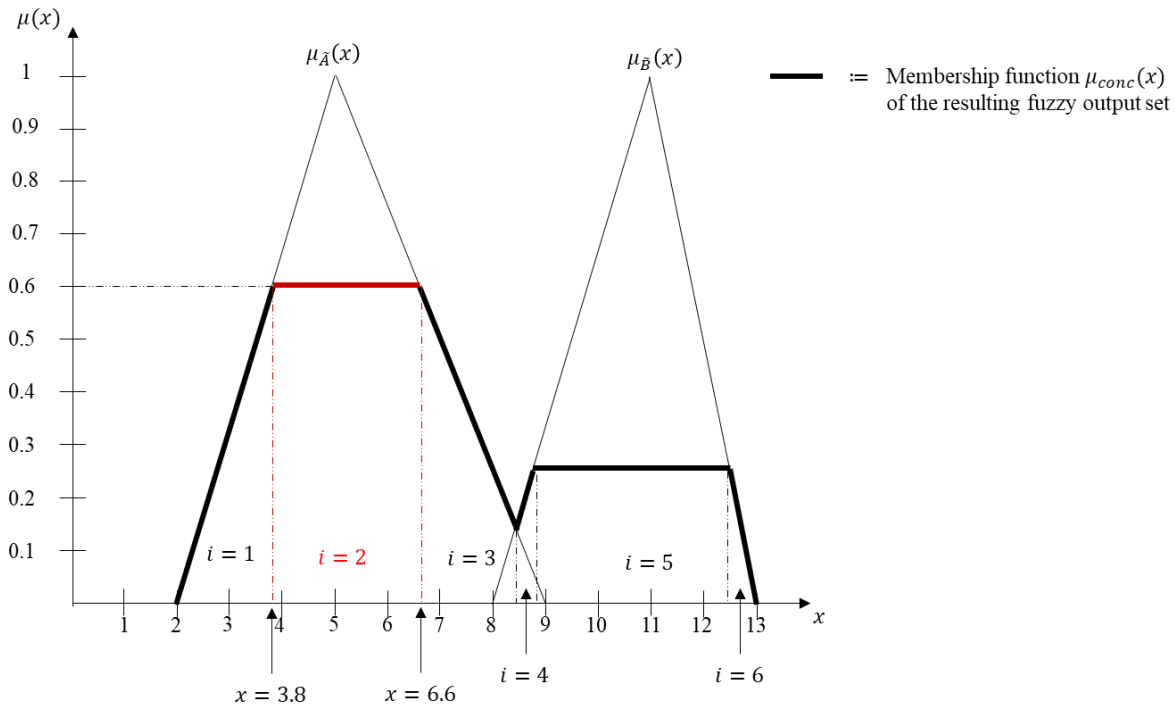


Figure 16: Consideration of section $i = 2$

$$x_{cog}^{i=2} = \frac{\int_{3.8}^{6.6} x \cdot 0.6\, dx}{\int_{3.8}^{6.6} 0.6\, dx} = \frac{0.3x^2|_{3.8}^{6.6}}{0.6x|_{3.8}^{6.6}} = \frac{8.736}{1.68}$$

Section $i = 3$:

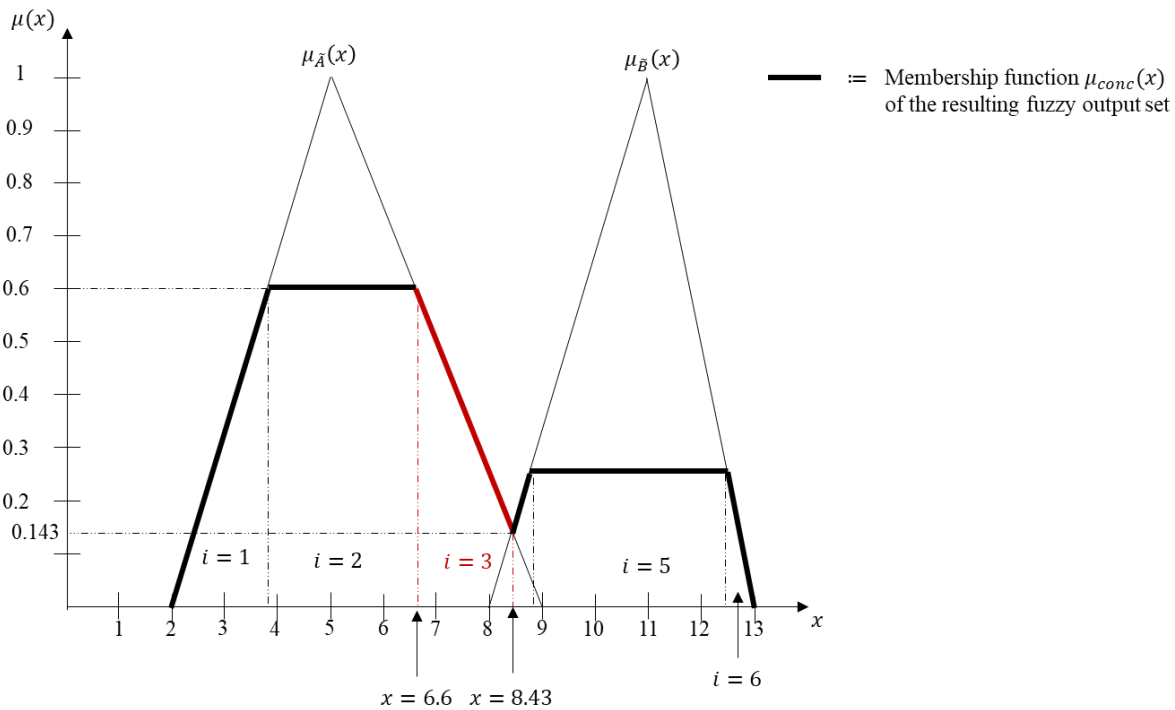


Figure 17: Consideration of section $i = 3$

$$x_{cog}^{i=3} = \frac{\int_{6.6}^{8.43} x \cdot (\frac{9}{4} - \frac{1}{4}x)\, dx}{\int_{6.6}^{8.43} (\frac{9}{4} - \frac{1}{4}x)\, dx} = \frac{1.125x^2 - \frac{1}{12}x^3|_{6.6}^{8.43}}{\frac{9}{4}x - \frac{1}{8}x^2|_{6.6}^{8.43}} = \frac{4.978}{0.679}$$

Section $i = 4$:

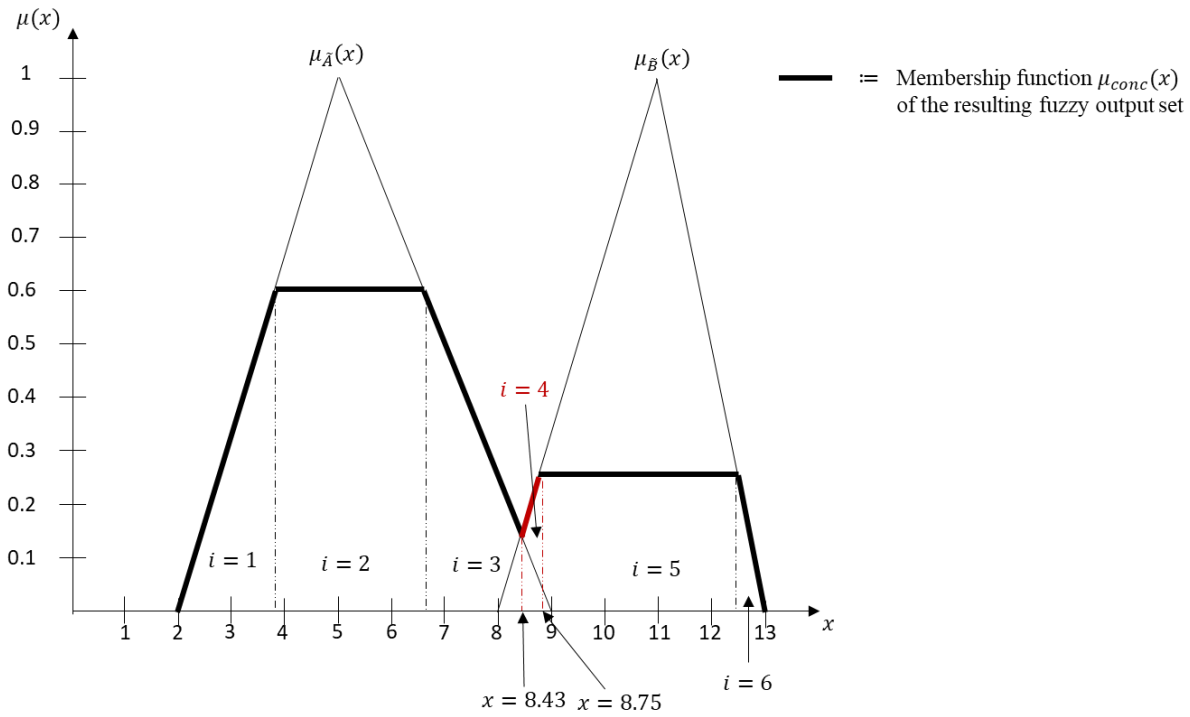


Figure 18: Consideration of section $i = 4$

$$x_{cog}^{i=4} = \frac{\int_{8.43}^{8.75} x \cdot (-\frac{8}{3} + \frac{1}{3}x)\, dx}{\int_{8.43}^{8.75} (-\frac{8}{3} + \frac{1}{3}x)\, dx} = \frac{-\frac{4}{3}x^2 + \frac{1}{9}x^3|_{8.43}^{8.75}}{-\frac{8}{3}x + \frac{1}{6}x^2|_{8.43}^{8.75}} = \frac{0.542}{0.063}$$

Section $i = 5$:

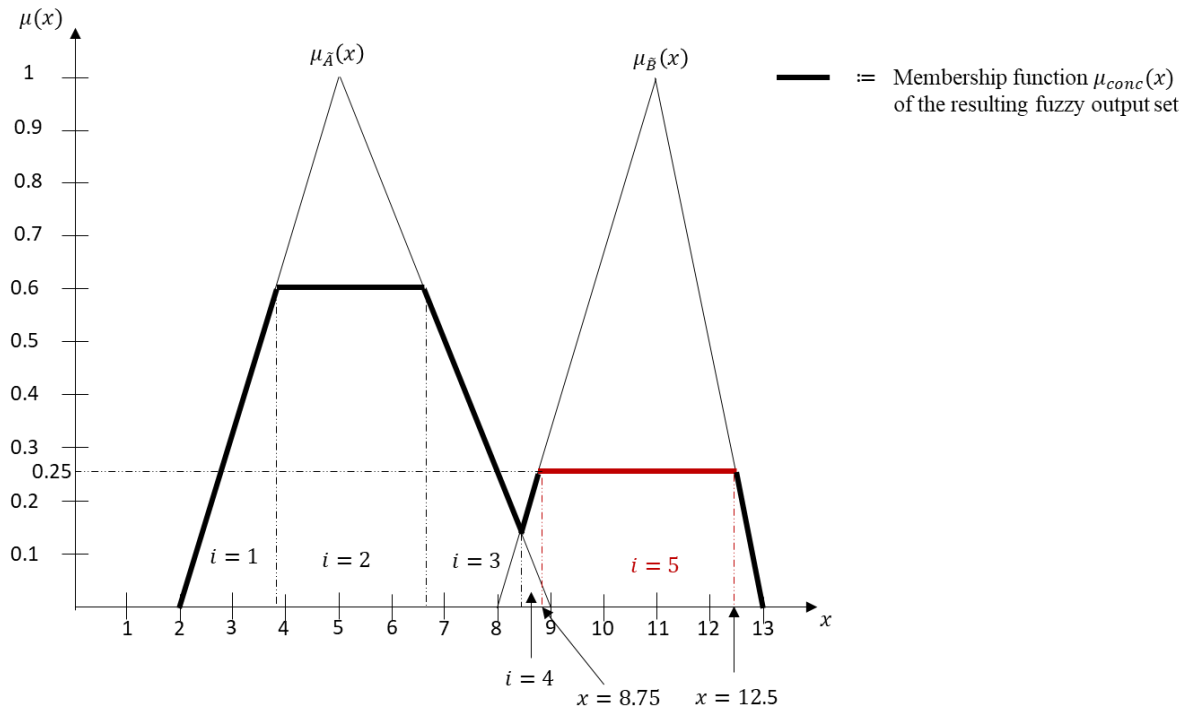


Figure 19: Consideration of section $i = 5$

$$x_{cog}^{i=5} = \frac{\int_{8.75}^{12.5} x \cdot 0.25\, dx}{\int_{8.75}^{12.5} 0.25\, dx} = \frac{0.125x^2|_{8.75}^{12.5}}{0.25x|_{8.75}^{12.5}} = \frac{9.961}{0.938}$$

Section $i = 6$:

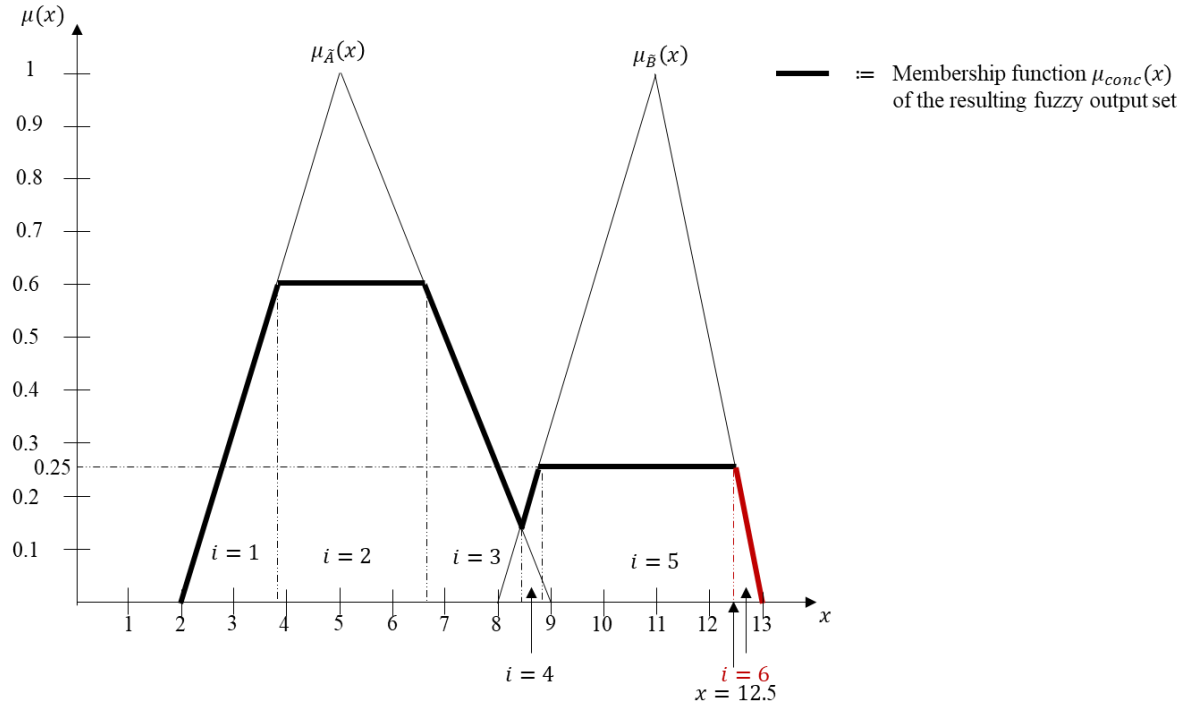


Figure 20: Consideration of section $i = 6$

$$x_{cog}^{i=6} = \frac{\int_{12.5}^{13} x \cdot (\frac{13}{2} - \frac{1}{2}x)\, dx}{\int_{12.5}^{13} (\frac{13}{2} - \frac{1}{2}x)\, dx} = \frac{3.25x^2 - \frac{1}{6}x^3|_{12.5}^{13}}{6.5x - \frac{1}{4}x^2|_{12.5}^{13}} = \frac{0.792}{0.063}$$

The additive combination of the expressions then leads to the determination of the abscissa coordinate of the centroid x_{cog} of the area of the fuzzy output set. Using the aggregation rule (10.3) already described above, we obtain:

$$x_{cog} = \frac{\sum_{i=1}^{6} \left[\int_{wlb}^{wub} x \cdot \mu_{conc}(x) dx\right]_i}{\sum_{i=1}^{6} \left[\int_{wlb}^{wub} \mu_{conc}(x) dx\right]_i} =$$

$$= \frac{1.728 + 8.736 + 4.978 + 0.542 + 9.961 + 0.792}{0.54 + 1.68 + 0.679 + 0.063 + 0.938 + 0.063} = \frac{26.737}{3.963} \approx 6.75$$

Following the determination of the abscissa coordinate of the centroid of the area, the ordinate coordinate can now be determined.

3.2.2 Determination of the ordinate coordinate of the centroid of the area

The general equation of determination (5) for calculating the ordinate coordinate of a centroid of an area is:

$$y_{cog} = \frac{\int y dA}{\int dA} = \frac{\int y\, x\, dy}{\int x\, dy} \tag{5}$$

To apply this determination equation, the inverse functions of $\mu_{conc}(x)$ must first be formed for relevant partial integrals.

Recalling the use of a notation in terms of membership functions, this then leads to the following determinant equation:

$$\mu(x_{cog}^{i}) = \frac{\int_{hlb}^{hub} \mu(x) \cdot \mu_{conc}^{-1}(x)\, d\mu(x)}{\int_{hlb}^{hub} \mu_{conc}^{-1}(x) d\mu(x)} \quad \forall i \in \bar{I} \tag{11.1}$$

Note:

The pure consideration of the inverse function would lead to the integration of the area between ordinate and function (dotted, red area in figure 21). However, for the determination of the ordinate coordinate of the centroid of the area, the consideration of the area between two function areas is relevant. This means that for section $i = 1$, the vertically hatched area is of interest, but not the area marked in red (see figure 21). Accordingly, when using the inverse function,

care must be taken to use it only for the area up to $x = 3.8$. This is achieved by subtracting the inverse function from the "limiting function".

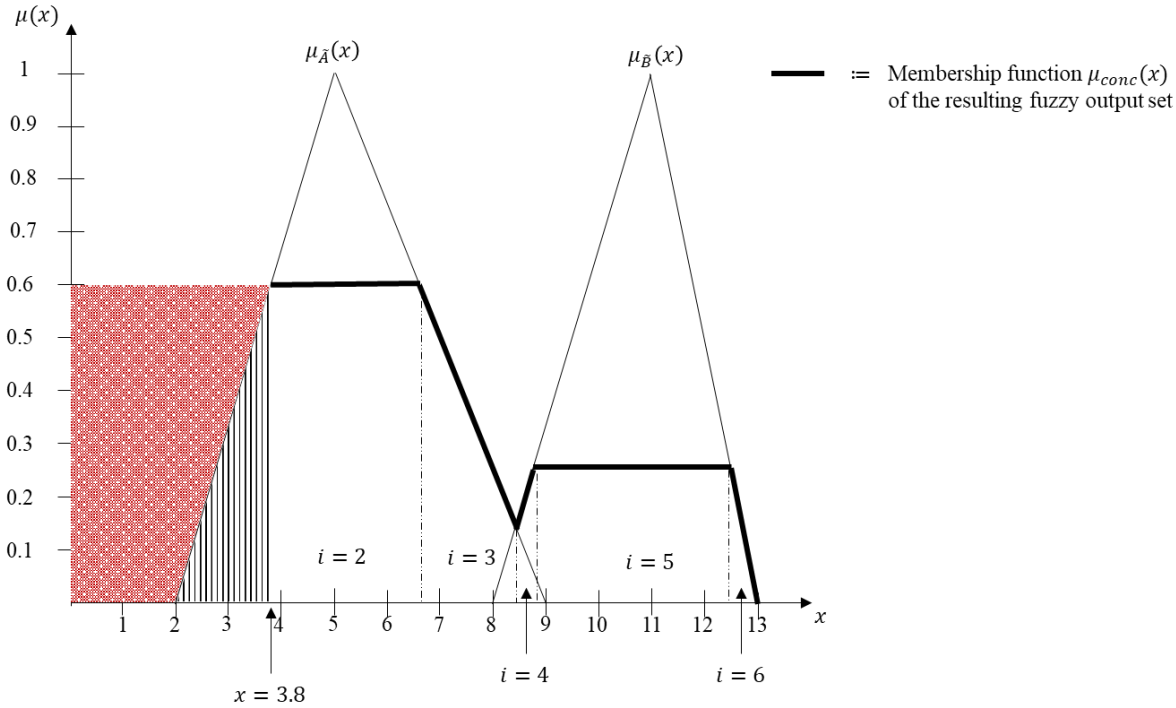


Figure 21: Note on the ordinate determination of a centroid of an area

Section $i = 1$:

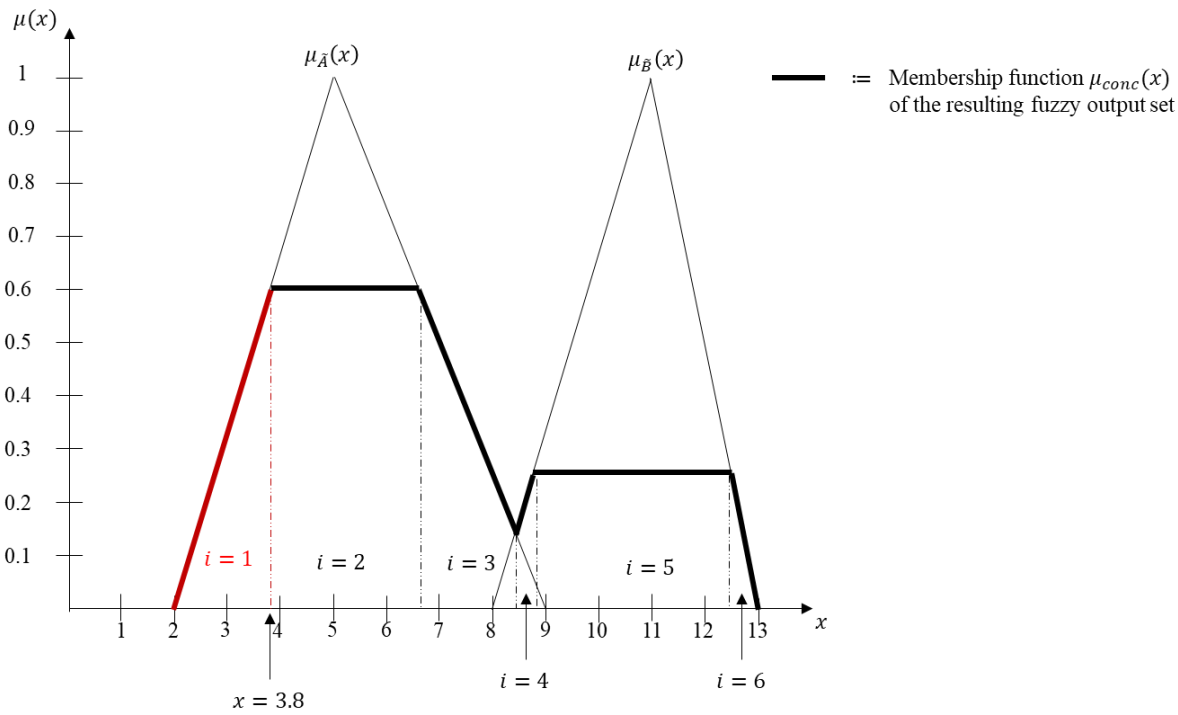


Figure 22: Consideration of section $i = 1$

Origin function:

$$\mu_{conc}(x) = -\frac{2}{3} + \frac{1}{3}x$$

Inverse function:

$$\mu_{conc}^{-1}(x) = 3 \cdot \mu(x) + 2$$

Limitation at $x = 3.8$ leads to:

$$\mu_{conc}^{-1}(x) = 3.8 - (3 \cdot \mu(x) + 2)$$
$$= 1.8 - 3 \cdot \mu(x)$$

$$\mu\left(x_{cog}^{i=1}\right) = \frac{\int_0^{0.6} \mu(x) \cdot \mu_{conc}^{-1}(x)\, d\mu(x)}{\int_0^{0.6} \mu_{conc}^{-1}(x)\, d\mu(x)} = \frac{\int_0^{0.6} \mu(x) \cdot \left(1.8 - 3 \cdot \mu(x)\right) d\mu(x)}{\int_0^{0.6} \left(1.8 - 3 \cdot \mu(x)\right) d\mu(x)}$$

$$= \frac{\int_0^{0.6} \left(-3 \cdot \mu(x)^2 + 1.8 \cdot \mu(x)\right) d\mu(x)}{\int_0^{0.6} (-3 \cdot \mu(x) + 1.8)\, d\mu(x)} = \frac{-\mu(x)^3 + 0.9\mu(x)^2|_0^{0.6}}{-\frac{3}{2}\mu(x)^2 + 1.8\mu(x)|_0^{0.6}}$$

$$= \frac{-0.6^3 + 0.9 \cdot 0.6^2}{-\frac{3}{2} \cdot 0.6^2 + 1.8 \cdot 0.6} = \frac{0.108}{0.54}$$

Section $i = 2$:

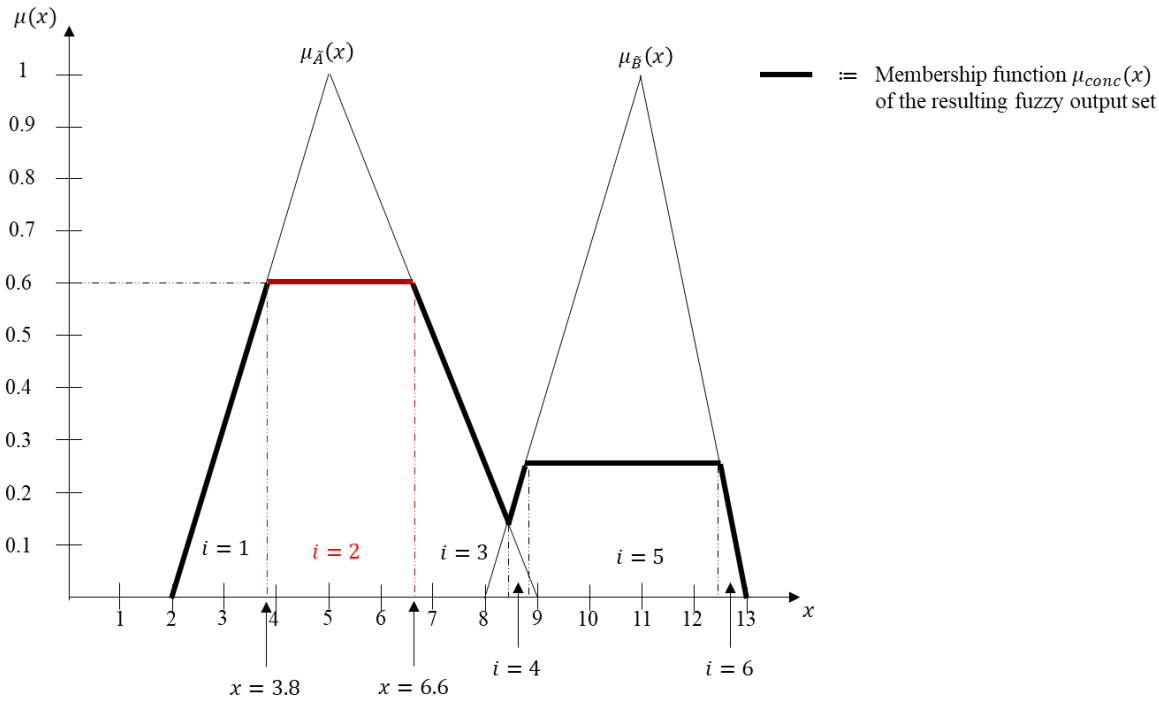


Figure 23: Consideration of section $i = 2$

The formation of an inverse function is not necessary in this section. Using the simplification described above in the case of rectangles with constant widths and heights, the ordinate coordinate $\mu\left(x_{cog}^{i=2}\right)$ can be determined as follows:

$$\mu\left(x_{cog}^{i=2}\right) = \frac{\int \mu(x)\, dA}{\int dA} = \frac{\int \mu(x) \cdot \mu(x_{conc}^{-1})\, d\mu(x)}{\int x\, d\mu(x)} = \frac{\int \mu(x) \cdot (wub - wlb) d\mu(x)}{\int (wub - wlb)\, d\mu(x)}$$

With $(wub - wlb)$ as the width of the rectangle and thus $(wub - wlb) = 6.6 - 3.8 = 2.8$ and $(hub - hlb)$ as the height of the rectangle and thus $(hub - hlb) = 0.6 - 0$ holds:

$$\mu\left(x_{cog}^{i=2}\right) = \frac{\int \mu(x) \cdot 2.8 \, d\mu(x)}{\int 2.8 \, d\mu(x)} = \frac{1.4\mu(x)^2|_0^{0.6}}{2.8\mu(x)\,|_0^{0.6}} = \frac{1.4 \cdot 0.6^2}{2.8 \cdot 0.6} = \frac{0.504}{1.68}$$

Section $i = 3$:

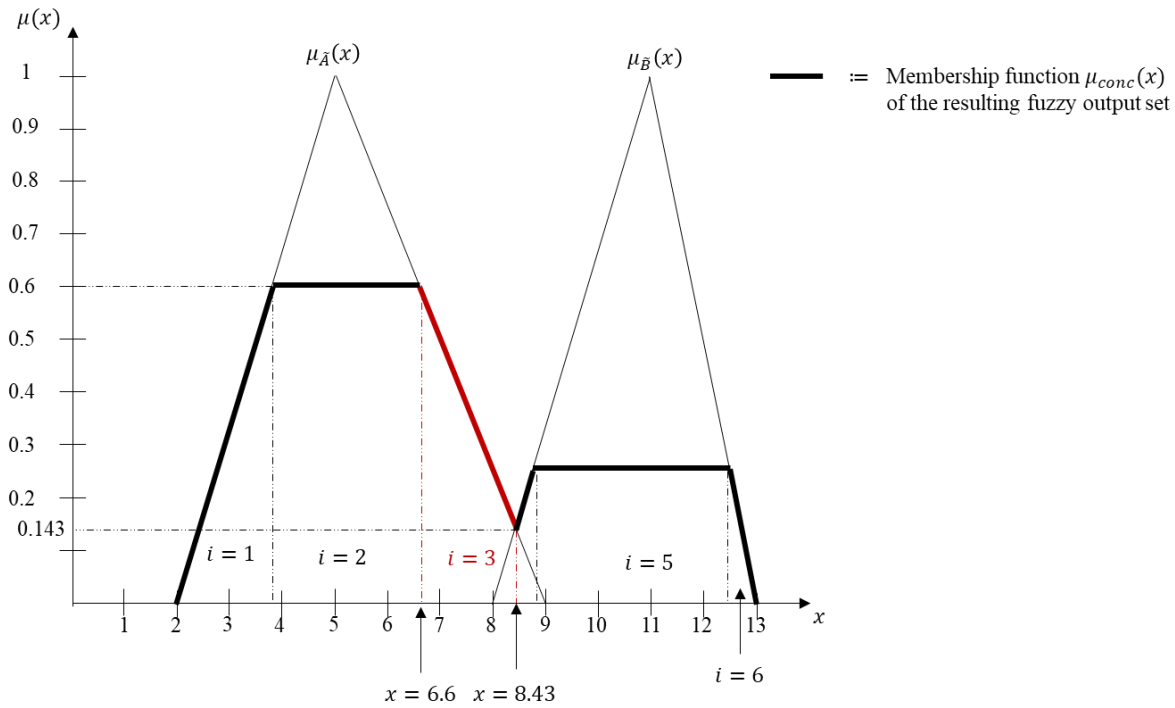


Figure 24: Consideration of section $i = 3$

Section $i = 3$ is to be divided into two parts (triangle and rectangle).

Accordingly, both a triangle ($A_{i=3|1}$; vertically hatched area in figure 25) and a rectangle ($A_{i=3|2}$; tiled area in figure 25) result as partial areas of the section.

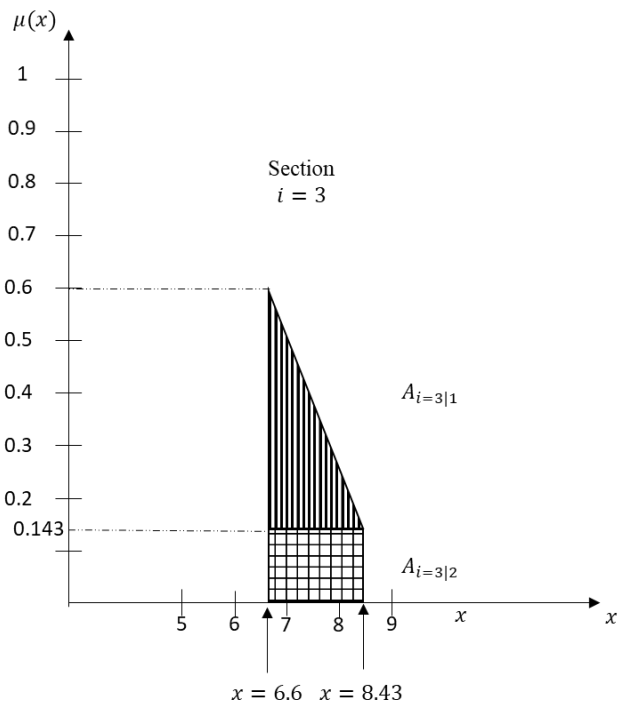


Figure 25: Decomposition of section $i = 3$ in partial areas

For the boundary of the triangle, this encloses the area between $x = 6.6$ and $x = 8.43$. Accordingly, the following modification of the inverse function results.

Origin function:

$$\mu_{conc}(x) = \frac{9}{4} - \frac{1}{4}x$$

Inverse function:

$$\mu_{conc}^{-1}(x) = 9 - 4\mu(x)$$

Limitation at $x = 6.6$ leads to:

$$\mu_{conc}^{-1}(x) = 9 - 4\mu(x) - 6.6$$
$$= 2.4 - 4\mu(x)$$

Determination of the ordinate coordinate of the centroid of the triangle's area $A_{i=3|1}$:

$$\mu\left(x_{cog}^{i=3|1}\right) = \frac{\int_{\frac{1}{7}}^{0.6} \mu(x) \cdot \mu_{conc}^{-1}(x)\, d\mu(x)}{\int_{\frac{1}{7}}^{0.6} \mu_{conc}^{-1}(x)\, d\mu(x)} = \frac{\int_{\frac{1}{7}}^{0.6} \mu(x) \cdot \left(2.4 - 4\mu(x)\right) d\mu(x)}{\int_{\frac{1}{7}}^{0.6} \left(2.4 - 4\mu(x)\right) d\mu(x)}$$

$$= \frac{\int_{\frac{1}{7}}^{0.6} (2.4\mu(x) - 4\mu(x)^2) d\mu(x)}{\int_{\frac{1}{7}}^{0.6} (2.4 - 4\mu(x))\, d\mu(x)} = \frac{1.2\mu(x)^2 - \frac{4}{3}\mu(x)^3 |_{\frac{1}{7}}^{0.6}}{2.4\mu(x) - 2\mu(x)^2 |_{\frac{1}{7}}^{0.6}} = \frac{0.123}{0.418}$$

Determination of the ordinate coordinate of the center of area of the rectangle $A_{i=3|2}$:

The formation of an inverse function is not necessary in this section. Using the simplification described above in the case of rectangles with constant widths and heights, the ordinate coordinate of the rectangle $\mu\left(x_{cog}^{i=3}\right)$ can be determined as follows:

With $(wub - wlb)$ as the width of the rectangle and thus $(wub - wlb) = 8.43 - 6.6 = 1.83$ and $(hub - hlb)$ as the height of the rectangle and thus $(hub - hlb) = 1/7 - 0$ holds:

$$\mu\left(x_{cog}^{i=3|2}\right) = \frac{\int \mu(x) \cdot 1.83 \, d\mu(x)}{\int 1.83 \, d\mu(x)} = \frac{0.915\mu(x)^2 \Big|_0^{\frac{1}{7}}}{1.83\mu(x) \Big|_0^{\frac{1}{7}}} = \frac{\frac{183}{9800}}{0.2614} = \frac{0.019}{0.261}$$

Aggregated, the ordinate coordinate of the centroid of area of section $i = 3$ is thus valid:

$$\mu\left(x_{cog}^{i=3}\right) = \frac{0.123 + 0.019}{0.418 + 0.261} = \frac{0.142}{0.679}$$

Section $\boldsymbol{i = 4}$:

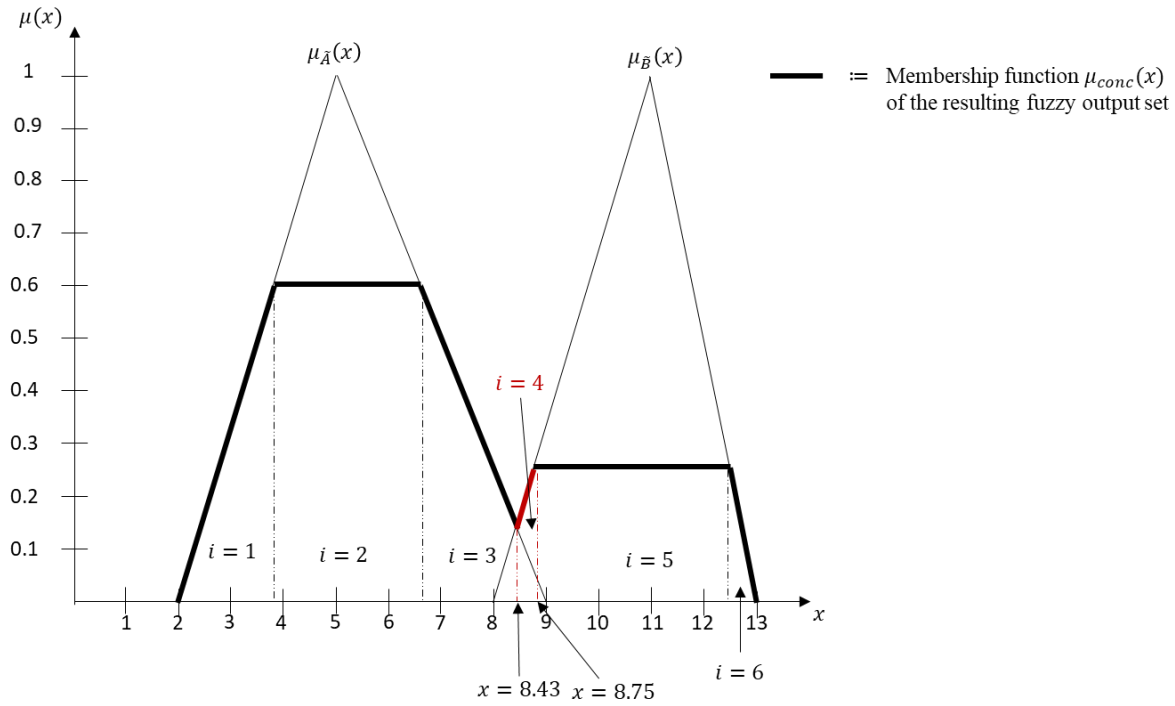


Figure 26: Consideration of section $i = 4$

Section $i = 4$ is also to be divided into two parts (triangle $A_{i=4|1}$ and rectangle $A_{i=4|2}$) in analogy to section $i = 3$.

For the boundary of the triangle it is valid that this encloses the area between $x = 8.43$ and $x = 8.75$.

Accordingly the following modification of the inverse function results.

Origin function:

$$\mu_{conc}(x) = -\frac{8}{3} + \frac{1}{3}x$$

Inverse function:

$$\mu_{conc}^{-1}(x) = 8 + 3\mu(x)$$

Limitation at $x = 8.75$ leads to:

$$\mu_{conc}^{-1}(x) = 8.75 - (3\mu(x) + 8)$$
$$= 0.75 - 3\mu(x)$$

Determination of the ordinate coordinate of the centroid of the triangle's area $A_{i=4|1}$:

$$\mu\left(x_{cog}^{i=4|1}\right) = \frac{\int_{\frac{1}{7}}^{0.25} \mu(x) \cdot \mu_{conc}^{-1}(x)\, d\mu(x)}{\int_{\frac{1}{7}}^{0.25} \mu_{conc}^{-1}(x)\, d\mu(x)} = \frac{\int_{\frac{1}{7}}^{0.25} \mu(x) \cdot \left(0.75 - 3\mu(x)\right) d\mu(x)}{\int_{\frac{1}{7}}^{0.25} \left(0.75 - 3\mu(x)\right) d\mu(x)}$$

$$= \frac{\int_{\frac{1}{7}}^{0.25} (0.75\mu(x) - 3\mu(x)^2) d\mu(x)}{\int_{\frac{1}{7}}^{0.25} (0.75 - 3\mu(x))\, d\mu(x)} = \frac{0.375\mu(x)^2 - \mu(x)^3 |_{\frac{1}{7}}^{0.25}}{0.75\mu(x) - 1.5\mu(x)^2 |_{\frac{1}{7}}^{0.25}} = \frac{0.003}{0.017}$$

Determination of the ordinate coordinate of the center of area of the rectangle $A_{i=4|2}$:

The formation of an inverse function is not necessary in this section. Using the simplification described above in the case of rectangles with constant widths and heights, the ordinate coordinate of the rectangle $\mu\left(x_{cog}^{i=4|2}\right)$ can be determined as follows.

With $(wub - wlb)$ as the width of the rectangle and thus $(wub - wlb) = 8.75 - 8.43 = 0.32$ and $(hub - hlb)$ as the height of the rectangle and thus $(hub - hlb) = 1/7 - 0$ holds:

$$\mu\left(x_{cog}^{i=4|2}\right) = \frac{\int \mu(x) \cdot 0.32\, d\mu(x)}{\int 0.32 d\mu(x)} = \frac{0.16\mu(x)^2 |_0^{\frac{1}{7}}}{0.32\mu(x)\, |_0^{\frac{1}{7}}} = \frac{\frac{4}{1225}}{\frac{8}{175}} = \frac{0.003}{0.046}$$

Aggregated, the ordinate coordinate of the centroid of area of section $i = 4$ is thus valid:

$$\mu\left(x_{cog}^{i=4}\right) = \frac{0.003 + 0.003}{0.017 + 0.046} = \frac{0.006}{0.063}$$

Section $i = 5$:

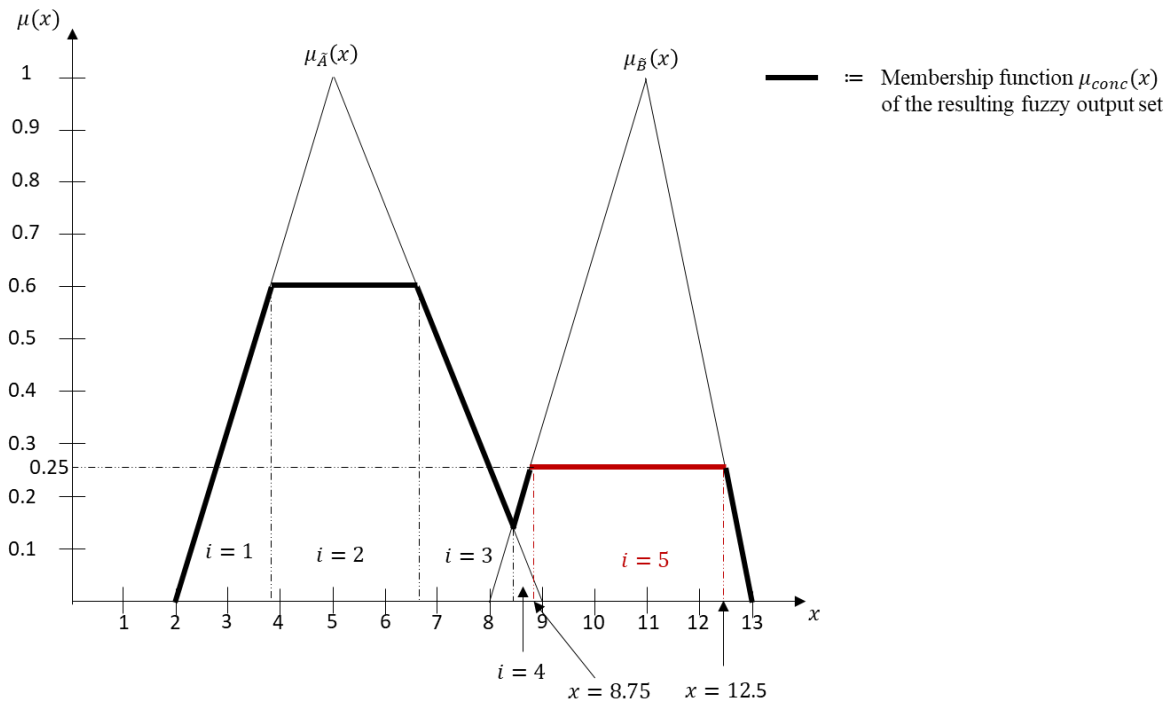


Figure 27: Consideration of section $i = 5$

The formation of an inverse function is also not necessary in this section. Using the simplification described above, with $(wub - wlb) = 12.5 - 8.75 = 3.75$ as the width and $(hub - hlb) = 0.25 - 0 = 0.25$ as the height of the rectangle, it applies:

$$\mu\left(x_{cog}^{i=5}\right) = \frac{\int \mu(x) \cdot 3.75 \, d\mu(x)}{\int 3.75 \, d\mu(x)} = \frac{1.875\mu(x)^2|_0^{0.25}}{3.75\mu(x)\,|_0^{0.25}} = \frac{1.875 \cdot 0.25^2}{3.75 \cdot 0.25} = \frac{0.117}{0.938}$$

Section $i = 6$:

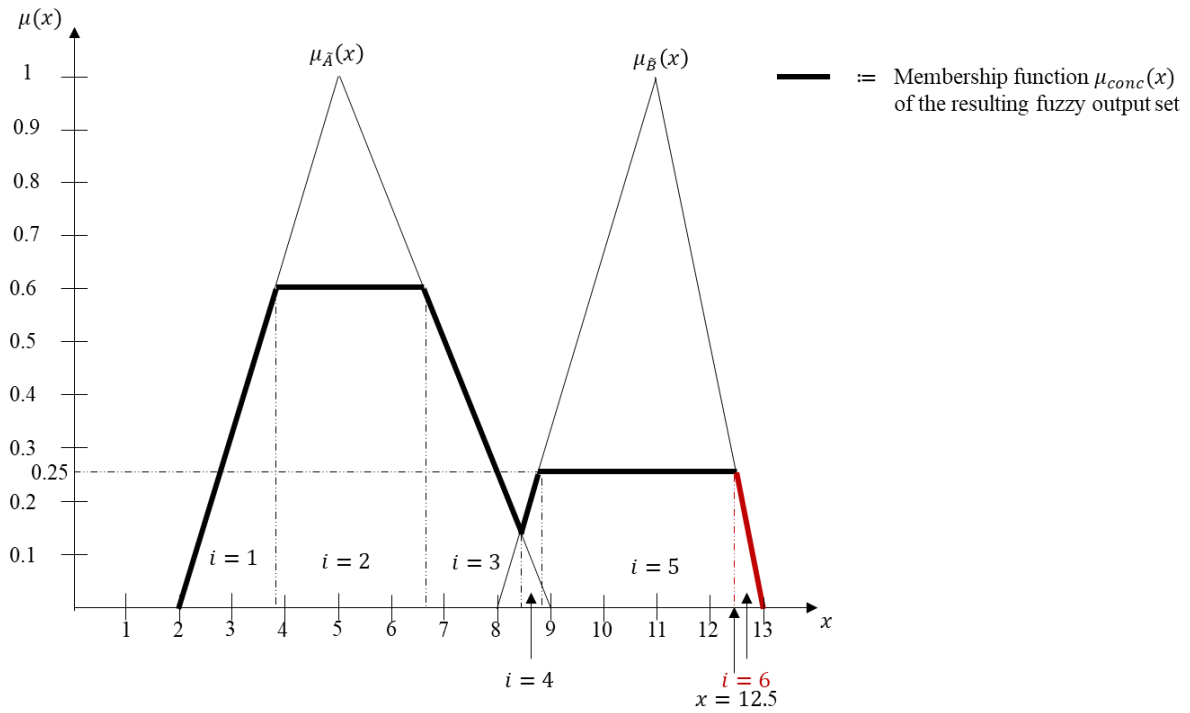


Figure 28: Consideration of section $i = 6$

Origin function:

$$\mu_{conc}(x) = \frac{13}{2} - \frac{1}{2}x$$

Inverse function:

$$\mu_{conc}^{-1}(x) = 13 - 2\mu(x)$$

Limitation at $x = 12.5$ leads to:

$$\mu_{conc}^{-1}(x) = (13 - 2\mu(x) - 12.5)$$
$$= 0.5 - 2\mu(x)$$

$$\mu\left(x_{cog}^{i=6}\right) = \frac{\int_0^{0.25} \mu(x) \cdot \mu_{conc}^{-1}(x)\, d\mu(x)}{\int_0^{0.25} \mu_{conc}^{-1}(x)\, d\mu(x)} = \frac{\int_0^{0.25} \mu(x) \cdot \left(0.5 - 2 \cdot \mu(x)\right) d\mu(x)}{\int_0^{0.25} \left(0.5 - 2 \cdot \mu(x)\right) d\mu(x)}$$

$$= \frac{\int_0^{0.25} (0.5\mu(x) - 2\mu(x)^2)\, d\mu(x)}{\int_0^{0.25} (0.5 - 2\mu(x))\, d\mu(x)} = \frac{-\frac{2}{3}\mu(x)^3 + 0.25\mu(x)^2|_0^{0.25}}{0.5\mu(x) - \mu(x)^2|_0^{0.25}} = \frac{0.005}{0.063}$$

The aggregation of the expressions leads to the determination of the ordinate coordinate of the centroid $\mu(x_{cog})$ of the area of the fuzzy output set:

$$\mu(x_{cog}) = \frac{\sum_{i=1}^{6}\left[\int_{hlb}^{hub} \mu(x) \cdot \mu_{conc}^{-1}(x) d\mu(x)\right]_i}{\sum_{i=1}^{6}\left[\int_{hlb}^{hub} \mu_{conc}^{-1}(x) d\mu(x)\right]_i} \tag{11.2}$$

In concrete terms, this means:

$$\mu(x_{cog}) = \frac{\sum_{i=1}^{6}\left[\int_{hlb}^{hub} \mu(x) \cdot \mu_{conc}^{-1}(x) d\mu(x)\right]_i}{\sum_{i=1}^{6}\left[\int_{hlb}^{hub} \mu_{conc}^{-1}(x) d\mu(x)\right]_i} =$$

$$= \frac{0.108 + 0.504 + 0.142 + 0.006 + 0.117 + 0.005}{0.54 + 1.68 + 0.679 + 0.063 + 0.938 + 0.063} = \frac{0.882}{3.963} \approx 0.222$$

In summary, this results in table 3:

Integral of section i	x_{cog}^{i}	Abscissa coordinate with isolated consideration of the partial integral in section i	$\mu(x_{cog}^{i})$	Ordinate coordinate when considering the partial integral in isolation in section i
1	$\frac{1.728}{0.54}$	3.2	$\frac{0.108}{0.54}$	0.2
2	$\frac{8.736}{1.68}$	5.2	$\frac{0.504}{1.68}$	0.3
3	$\frac{4.978}{0.679}$	7.33	$\frac{0.142}{0.6794}$	0.209
4	$\frac{0.542}{0.063}$	8.6	$\frac{0.006}{0.063}$	0.095
5	$\frac{9.961}{0.938}$	10.619	$\frac{0.117}{0.938}$	0.125
6	$\frac{0.792}{0.063}$	12.57	$\frac{0.005}{0.063}$	0.079

Table 3: Summary of the respective center of gravity coordinates of the partial integrals

Thus, the coordinates of the centroid of the area for the fuzzy output set shown above are: $x_{cog} \approx 6.75$ and $\mu(x_{cog}) \approx 0.222$.

The abscissa coordinate of the center of gravity x_{cog} can be interpreted as the "average" value of the abscissa expressions. Accordingly, this abscissa coordinate unites the "mass" of the ordinate expressions in one point. Analogously, $\mu(x_{cog})$ can be understood as the "mean" membership value of the fuzzy output set. Ultimately, the mass of abscissa expressions is concentrated here on one membership value. Thus, in the centroid, both the mass of membership values of the fuzzy output set and the mass of abscissa expressions are concentrated. At this point, the fuzzy output set is therefore in an equilibrium position (by analogy with statics).

We can insert these determined coordinates and the corresponding resulting centroid COG into the representation of the fuzzy output set for illustrative purposes (see figure 29).

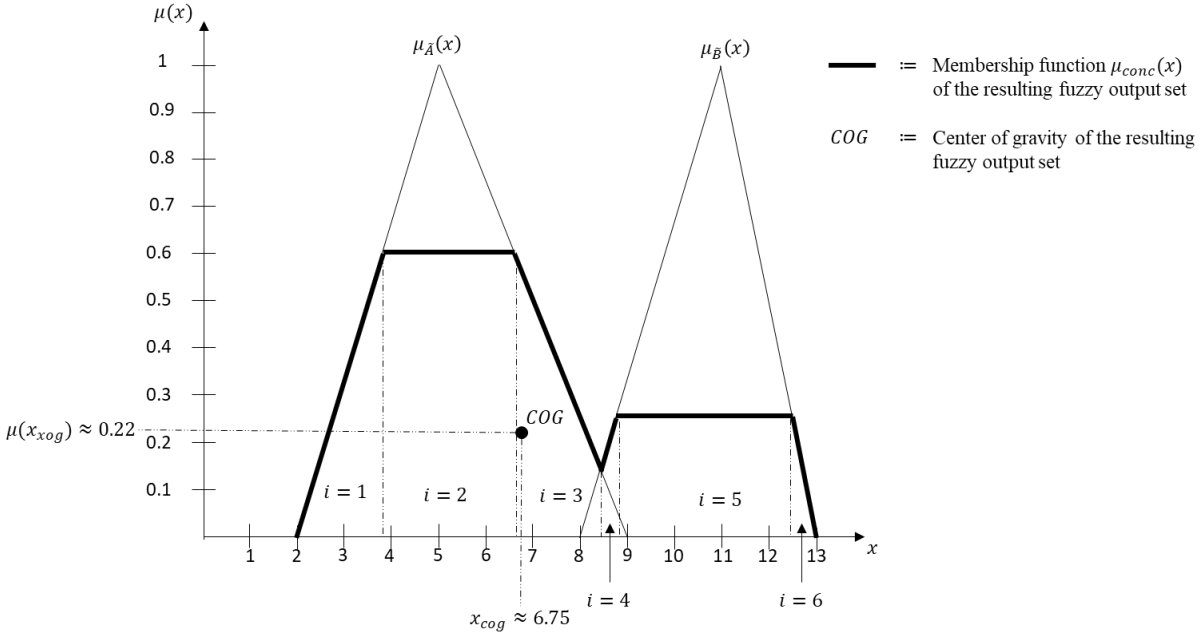


Figure 29: Graphical representation of membership functions with associated centroid of area of the fuzzy output set

3.3 Simplified centroid determination

In section 3.2, we focus on the application of the integral calculus for the determination of centroidal coordinates in order to show (in general) how to determine them for (at least piecewise) curved and for linear membership functions. Although we restrict ourselves in this contribution to the representation of linear membership function courses, the integral calculus has the advantage that it is applicable without modification also to function courses of non-linear form (e.g. power functions of higher order or root functions). When using purely linear function progressions, however, a simplified determination of the centroid of area can be carried out. With recourse to the above example the coordinates of the centroid can be determined as follows:

$$x_{cog} = \frac{\sum_{i=1}^{6} x_{cog}^{i} \cdot A_i}{\sum_{i=1}^{6} A_i} \tag{12}$$

$$\mu(x_{cog}) = \frac{\sum_{i=1}^{6} \mu(x_{cog}^{i}) \cdot A_i}{\sum_{i=1}^{6} A_i} \tag{13}$$

Here A_i corresponds to the area of the considered subarea i and x_{cog}^{i} respectively $\mu(x_i^{cog})$ to the abscissa respectively ordinate coordinate of the centroid of section i.

It is not difficult to see that this form of representation corresponds only to a modified form of notation compared to the above representation of the integral notation. The simplification results from the fact that we do not fall back now for the determination of the area centroid coordinates on the integral calculation, but use rather simple equations for the determination of area contents and area centroids.

We will now illustrate this simplified procedure for determining the centroid of an area with reference to the above example:

Section $\boldsymbol{i = 1}$

Determination of the area of the triangular area $A_{i=1}$

$$A_{i=1} = \frac{1}{2} \cdot (wub - wlb) \cdot (hub - hlb) = \frac{1}{2} \cdot (3.8 - 2) \cdot 0.6 = 0.54$$

Determination of the centroid coordinates of the area $A_{i=1}$

To determine the centroid of the area of a right triangle and the coordinates of three vertices A, B, and C $[(x_A|y_A), (x_B|y_B), (x_C|y_C)]$, we can refer to the following determination rule in our notation:

$$\left(x_{cog}^{i=1} \middle| \mu(x)_{cog}^{i=1}\right) = \frac{1}{3} \cdot \begin{pmatrix} x_A^{i=1} + x_B^{i=1} + x_C^{i=1} \\ \mu\left(x_A^{i=1}\right) + \mu\left(x_B^{i=1}\right) + \mu\left(x_C^{i=1}\right) \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 2 + 3.8 + 3.8 \\ 0 + 0 + 0.6 \end{pmatrix} = \begin{pmatrix} 3.2 \\ 0.2 \end{pmatrix}$$

Thus, for the coordinates of the centroid of the subarea $i = 1$ applies:

$x_{cog}^{i=1} = 3.2$ und $\mu(x)_{cog}^{i=1} = 0.2$

The derivation of the usability of this quite simple determination of the centroid coordinates of a triangle we want to present briefly for didactic reasons. For this purpose we will use a triangle with the three coordinates $A(2|0), B(8|0)$, and $C(4|1)$ (see figure 30):

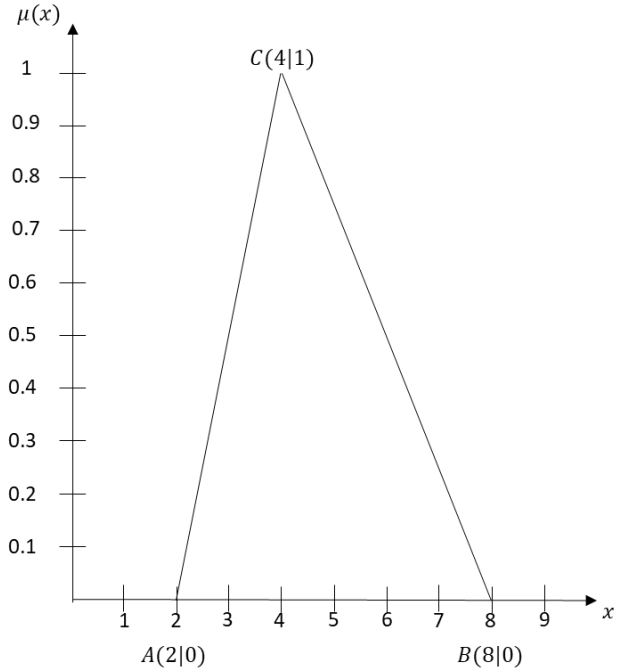


Figure 30: Graphical representation of a triangle with coordinates A, B, and C

The coordinates of the triangle can be represented as location vectors as follows:

$$\overrightarrow{OA} = \vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OB} = \vec{b} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OC} = \vec{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The center of area COG of a triangle results from the intersection of the side bisectors (see figure 31)

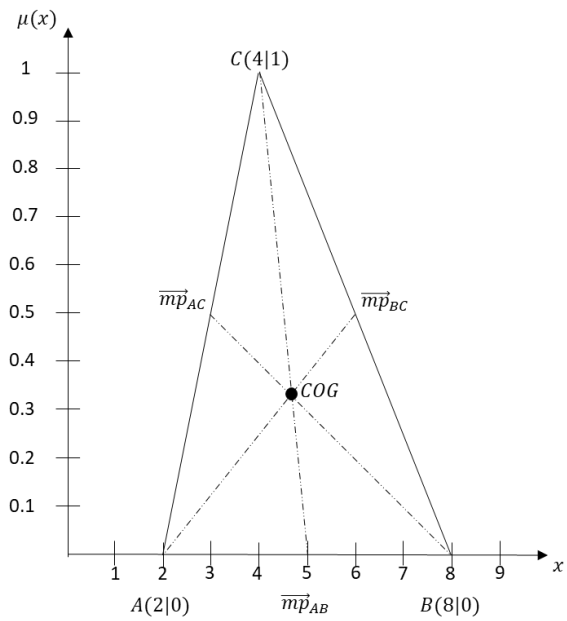


Figure 31: Graphical representation of the bisectors of the sides of a triangle

For $\overrightarrow{mp}_{AB}$, $\overrightarrow{mp}_{AC}$ and $\overrightarrow{mp}_{BC}$ as vectors of the center coordinates of the side bisectors holds:

$$\overrightarrow{mp}_{AB} = \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2} \cdot \left[\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix}\right] = \begin{pmatrix}5\\0\end{pmatrix}$$

$$\overrightarrow{mp}_{AC} = \frac{1}{2}(\vec{a} + \vec{c}) = \frac{1}{2} \cdot \left[\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix}\right] = \begin{pmatrix}3\\0.5\end{pmatrix}$$

$$\overrightarrow{mp}_{BC} = \frac{1}{2}(\vec{b} + \vec{c}) = \frac{1}{2} \cdot \left[\begin{pmatrix}8\\0\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix}\right] = \begin{pmatrix}6\\0.5\end{pmatrix}$$

For subsequent determination equations we only use $\overrightarrow{mp}_{AB}$. An analogous procedure can also be carried out with the vectors $\overrightarrow{mp}_{AC}$ as well as $\overrightarrow{mp}_{BC}$. Of interest now is the determination of the vector from $\overrightarrow{mp}_{AB}$ to the point C:

$$\overrightarrow{mp_{AB}C} = \vec{c} - \overrightarrow{mp}_{AB} = \begin{pmatrix}4\\1\end{pmatrix} - \begin{pmatrix}5\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

The center of gravity COG is always closer to the respective reference edge of the side bisector than to the corresponding corner point. The length of the leg that runs to the reference edge is $\frac{1}{3}$ and that to the corner point is $\frac{2}{3}$ of the total length of the bisector.

Based on this, the vector of the center of gravity $\overrightarrow{sp}$ can be determined as follows:

$$\overrightarrow{sp} = \overrightarrow{mp}_{AB} + \frac{1}{3} \cdot \overrightarrow{mp_{AB}C} = \begin{pmatrix}5\\0\end{pmatrix} + \frac{1}{3} \cdot \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}4.\overline{66}\\0.\overline{33}\end{pmatrix}$$

Since $\overrightarrow{sp} = \overrightarrow{mp}_{AB} + \frac{1}{3} \cdot \overrightarrow{mp_{AB}C}$ holds, we can also do a slight transformation:

$$\overrightarrow{sp} = \overrightarrow{mp}_{AB} + \frac{1}{3} \cdot \overrightarrow{mp_{AB}C} = \frac{1}{2} \cdot (\vec{a} + \vec{b}) + \frac{1}{3} \cdot (\vec{c} - \overrightarrow{mp}_{AB})$$

Resulting from this

$$\overrightarrow{sp} = \frac{1}{2} \cdot (\vec{a} + \vec{b}) + \frac{1}{3}\left(\vec{c} - \left(\frac{1}{2}(\vec{a} + \vec{b})\right)\right)$$

und thus:

$$\overrightarrow{sp} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{3}\vec{c} - \frac{1}{6}\vec{a} - \frac{1}{6}\vec{b}$$

und with this:

$$\overrightarrow{sp} = \frac{1}{3} \cdot (\vec{a} + \vec{b} + \vec{c})$$

This corresponds to the separate addition of the abscissa and ordinate coordinates of a triangle and the subsequent multiplication by $\frac{1}{3}$.

Section $i = 2$

Determination of the area of the rectangular area $A_{i=2}$

$$A_{i=2} = (wub - wlb) \cdot (hub - hlb) = (6.6 - 3.8) \cdot (0.6 - 0) = 1.68$$

Determination of the centroid coordinates of the area $A_{i=2}$

$$x_{cog}^{i=2} = \frac{1}{2} \cdot (wub - wlb) + wlb = \frac{1}{2} \cdot (6.6 - 3.8) + 3.8 = 5.2$$

$$\mu(x)_{cog}^{i=2} = \frac{1}{2} \cdot (hub - hlb) = \frac{1}{2} \cdot (0.6 - 0) = 0.3$$

Thus, for the coordinates of the centroid of the subarea $i = 2$, it applies:

$x_{cog}^{i=2} = 5.2$ and $\mu(x)_{cog}^{i=2} = 0.3$

Section $i = 3$

In section $i = 3$, there is the further specificity that this section must be divided into two sub-sections. Accordingly, both a triangle ($A_{i=3|1}$; vertically hatched area in figure 25) and a rectangle ($A_{i=3|2}$; tiled area in figure 25) result as partial areas of the section.

Determination of the area of the triangular area $A_{i=3|1}$

$$A_{i=3|1} = \frac{1}{2} \cdot (wub - wlb) \cdot (hub - hlb) = \frac{1}{2}(8.43 - 6.6) \cdot \left(0.6 - \frac{1}{7}\right) = \frac{366}{875} = 0.418$$

Determination of the area of the rectangular area $A_{i=3|2}$

$$A_{i=3|2} = (wub - wlb) \cdot (hub - hlb) = (8.43 - 6.6) \cdot \left(\frac{1}{7} - 0\right) = 0.261$$

Determination of the centroid coordinates of the area $A_{i=3|1}$

$$\left(x_{cog}^{i=3|1} \middle| \mu(x)_{cog}^{i=3|1}\right) = \frac{1}{3} \cdot \begin{pmatrix} x_A^{i=3|1} + x_B^{i=3|1} + x_C^{i=3|1} \\ \mu\left(x_A^{i=3|1}\right) + \mu\left(x_B^{i=3|1}\right) + \mu\left(x_C^{i=3|1}\right) \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 8.43 + 6.6 + 6.6 \\ \frac{1}{7} + \frac{1}{7} + 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 7.21 \\ \frac{31}{105} \end{pmatrix} = \begin{pmatrix} 7.21 \\ 0.295 \end{pmatrix}$$

Thus, for the coordinates of the centroid of the subarea $i = 3|1$, it applies:

$x_{cog}^{i=3|1} = 7.21$ and $\mu(x)_{cog}^{i=3|1} = \frac{31}{105} = 0.295$

Determination of the centroid coordinates of the area $A_{i=3|2}$

$$x_{cog}^{i=3|2} = \frac{1}{2} \cdot (wub - wlb) + wlb = \frac{1}{2} \cdot (8.43 - 6.6) + 6.6 = 7.515$$

$$\mu(x)_{cog}^{i=3|2} = \frac{1}{2} \cdot (hub - hlb) = \frac{1}{2} \cdot \left(\frac{1}{7} - 0\right) = \frac{1}{14} = 0.071$$

Section $i = 4$

For the determination of the centroid of area of section $i = 4$, an analogous procedure to section $i = 3$ is to be adopted.

Accordingly, both a triangle ($A_{i=4|1}$) and a rectangle ($A_{i=4|2}$) result as partial areas of the section.

Determination of the area of the triangular area $A_{i=4|1}$

$$A_{i=4|1} = \frac{1}{2} \cdot (wub - wlb) \cdot (hub - hlb) = \frac{1}{2}(8.75 - 8.43) \cdot \left(0.25 - \frac{1}{7}\right) = \frac{3}{175} = 0.017$$

Determination of the area of the rectangular area $A_{i=4|2}$

$$A_{i=4|2} = (wub - wlb) \cdot (hub - hlb) = (8.75 - 8.43) \cdot \left(\frac{1}{7} - 0\right) = \frac{8}{175} = 0.046$$

Determination of the centroid coordinates of the area $A_{i=4|1}$

$$\left(x_{cog}^{i=4|1} \middle| \mu(x)_{cog}^{i=4|1}\right) = \frac{1}{3} \cdot \begin{pmatrix} x_A^{i=4|1} + x_B^{i=4|1} + x_C^{i=4|1} \\ \mu\left(x_A^{i=4|1}\right) + \mu\left(x_B^{i=4|1}\right) + \mu\left(x_C^{i=4|1}\right) \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 8.43 + 8.75 + 8.75 \\ \frac{1}{7} + 0.25 + \frac{1}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 8.643 \\ \frac{5}{28} \end{pmatrix} = \begin{pmatrix} 8.643 \\ 0.179 \end{pmatrix}$$

Thus, for the coordinates of the centroid of the subarea $i = 4|1$ holds:

$x_{cog}^{i=4|1} = 8.643$ and $\mu(x)_{cog}^{i=4|1} = \frac{5}{28} = 0.179$

Determination of the centroid coordinates of the area $A_{i=4|2}$

$$x_{cog}^{i=4|2} = \frac{1}{2} \cdot (wub - wlb) + wlb = \frac{1}{2} \cdot (8.75 - 8.43) + 8.43 = 8.59$$

$$\mu(x)_{cog}^{i=4|2} = \frac{1}{2} \cdot (hub - hlb) = \frac{1}{2} \cdot \left(\frac{1}{7} - 0\right) = \frac{1}{14} = 0.071$$

Section $i = 5$

Determination of the area of the rectangular area $A_{i=5}$

$$A_{i=5} = (wub - wlb) \cdot (hub - hlb) = (12.5 - 8.75) \cdot (0.25 - 0) = 0.938$$

Determination of the centroid coordinates of the area $A_{i=5}$

$$x_{cog}^{i=5} = \frac{1}{2} \cdot (wub - wlb) + wlb = \frac{1}{2} \cdot (12.5 - 8.75) + 8.75 = 10.625$$

$$\mu(x)_{cog}^{i=5} = \frac{1}{2} \cdot (hub - hlb) = \frac{1}{2} \cdot (0.25 - 0) = 0.125$$

Thus, for the coordinates of the centroid of the subarea $i = 5$ holds:

$x_{cog}^{i=5} = 10.625$ and $\mu(x)_{cog}^{i=5} = 0.125$

Section $i = 6$

Determination of the area of the triangular area $A_{i=6}$

$$A_{i=6} = \frac{1}{2} \cdot (wub - wlb) \cdot (hub - hlb) = \frac{1}{2} \cdot (13 - 12.5) \cdot (0.25 - 0) = 0.063$$

Determination of the centroid coordinates of the area $A_{i=6}$

To determine the centroid of the area of a right triangle and the coordinates of three vertices A, B, and C $[(x_A|y_A), (x_B|y_B), (x_C|y_C)]$, we can refer to the following determination rule in our notation:

$$\left(x_{cog}^{i=6} \middle| \mu(x)_{cog}^{i=6}\right) = \frac{1}{3} \cdot \begin{pmatrix} x_A^{i=6} + x_B^{i=6} + x_C^{i=6} \\ \mu\left(x_A^{i=6}\right) + \mu\left(x_B^{i=6}\right) + \mu\left(x_C^{i=6}\right) \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 12.5 + 13 + 12.5 \\ 0 + 0 + 0.25 \end{pmatrix}$$

$$= \begin{pmatrix} 12.667 \\ 0.083 \end{pmatrix}$$

Thus, for the coordinates of the centroid of the subarea $i = 6$ holds:

$x_{cog}^{i=6} = 12.667$ and $\mu(x)_{cog}^{i=6} = 0.083$

With the above determinations of the area contents and the respective area centroids, one can now determine the area centroid of the total area of the resulting fuzzy output set. Thus, for the abscissa coordinate of the centroid of the area x_{cog} holds:

$$x_{cog} = \frac{\sum_{i=1}^{6} x_{cog}^{i} \cdot A_i}{\sum_{i=1}^{6} A_i} =$$

$$= \frac{x_{cog}^{i=1} \cdot A_{i=1} + x_{cog}^{i=2} \cdot A_{i=2} + x_{cog}^{i=3|1} \cdot A_{i=3|1} + x_{cog}^{i=3|2} \cdot A_{i=3|2} + x_{cog}^{i=4|1} \cdot A_{i=4|1} + x_{cog}^{i=4|2} \cdot A_{i=4|2} + x_{cog}^{i=5} \cdot A_{i=5} + x_{cog}^{i=6} \cdot A_{i=6}}{A_{i=1} + A_{i=2} + A_{i=3|1} + A_{i=3|2} + A_{i=4|1} + A_{i=4|2} + A_{i=5} + A_{i=6}}$$

$$x_{cog} = \frac{3.2 \cdot 0.54 + 5.2 \cdot 1.68 + 7.21 \cdot 0.418 + 7.515 \cdot 0.261 + 8.643 \cdot 0.017 + 8.59 \cdot 0.046 + 10.625 \cdot 0.938 + 12.667 \cdot 0.063}{0.54 + 1.68 + 0.418 + 0.261 + 0.017 + 0.046 + 0.938 + 0.063}$$

$$x_{cog} = \frac{26.746}{3.963} = 6.749$$

In an analogous procedure, the following results for the ordinate coordinate $\mu(x_{cog})$ apply:

$$\mu(x_{cog}) = \frac{0.2 \cdot 0.54 + 0.3 \cdot 1.68 + 0.295 \cdot 0.418 + 0.071 \cdot 0.261 + 0.179 \cdot 0.017 + 0.071 \cdot 0.045 + 0.125 \cdot 0.938 + 0.083 \cdot 0.063}{0.54 + 1.68 + 0.418 + 0.261 + 0.017 + 0.046 + 0.938 + 0.063}$$

$$\mu(x_{cog}) = \frac{0.883}{3.963} = 0.223^{4}$$

Thus, to determine the values of x_{cog} and $\mu(x_{cog})$, we can omit complex integrals for purely triangular membership functions.

4 Summary

Besides the presentation of selected tools of scenario management (influence, consistency, cross-impact analysis), the subject of this paper is the presentation of various defuzzification methods in this field. We focus on the so-called maximum and center of gravity methods. The latter, which we also explain by way of example, is given a relatively large amount of space. It is shown that the determination of the center of area is very simple for triangular and trapezoidal membership functions of the fuzzy output. For curvilinear membership functions, on the other hand, one has to rely on the integral calculus. Moreover, we formulate an exemplary fuzzy rule system in the field of cross-impact analysis.

[4] The deviations between the determined ordinate coordinate and abscissa coordinate when using the simplified method and the determined ordinate coordinate and abscissa coordinate when using the integral calculation result from rounding differences.

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